UNCERTAINTY RESOLUTION IN MULTI-STAGE CAPITAL BUDGETING PROBLEMS

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Chan Seok Park

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UNCERTAINTY RESOLUTION IN MULTI-STAGE

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Approved:

Gerald J. Thuesen, Chairman

Gunter P. Sharp

Fred E. Williams

Date Approved by Chairman 3-11-77

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SUMMARY

The significance of the manner in which uncertainty is expected to be resolved over time in the capital rationing context is that it provides insight about changes in the level of uncertainty as the project's future cash flows become known. In a sequential decision process where future budgets are influenced by current decisions, the information about the proposal's level of uncertainty is quite useful.

Since the concept of uncertainty resolution reflects the changes in variability of economic worth of a proposal over time, a time-dependent measure of investment worth is developed. After an investigation of various time-dependent measures used to summarize the significant economic characteristics of an investment, the project balance is selected. The project balance is defined as the net equivalent amount the firm has invested in the project or has received from the project at the end of period t, if interest is compounded at a given interest rate.

The project balance pattern provides four different elements of information regarding the desirability of a proposal. They are the area of negative project balance (ANB), the discounted payback period (Q), the area of positive balance (APB), and the terminal profitability $S_N(i)$ of the proposal. Although each of these elements are important, only the negative project balance and the terminal profitability are used in the criterion developed.

To quantify uncertainty resolution, a measure of uncertainty

resolution based on the ANB with the expected gain confidence criterion [EGCL] is developed. The incorporation of the EGCL in the measure of uncertainty resolution enables the criterion to focus on the downside risks of loss as it changes over time.

An investment situation is suggested where investment decisions are made on a regular periodic basis and the objective is to maximize the total accumulated wealth at some horizon time. It is assumed that knowledge about what investment would be available in the future and their associated cash flows is probabilistic.

To provide a decision rule utilizing uncertainty resolution which can assess the effect of investment proposals that have probabilistic outcomes, a decision criterion called the Project Balance (PB) Criterion is developed. This criterion consists of a single index which seeks a practical trade-off among the three major investment factors: profitability, variability and flexibility.

A simulation model based on a variety of investment situations is developed to test the effectiveness of the PB criterion with other frequently mentioned decision criteria. These criteria are expected value maximization, the mean-variance (M-V) criterion, and expected utility maximization. In addition to comparing these three criteria with the PB criterion, the value of having complete information about the future investment opportunities is also introduced to compare the overall effectiveness of the PB criterion.

The analysis of the results indicate that the project balance criterion is generally superior to the other criteria investigated. This superiority increases when the variability of the outcomes increases and the cash-flow patterns are single payment rather than heterogeneous.

It is observed that the PB criterion is not sensitive to changes in the discount rate selected and to the choice of the risk aversion parameter, whereas the selection of the optimal interest rate is critical to the other decision criteria and the Mean-Variance criterion is very sensitive to the choice of the risk aversion parameter. Also, for the cases tested, the Project Balance criterion applied on a sequential basis is also seen to be just as effective as having perfect knowledge about the outcomes of only the projects known at the time of decision.

CHAPTER I

INTRODUCTION

Capital budgeting decisions have been one of the most important and most difficult decisions that have confronted businessmen for a number of decades. Since businessmen commonly have to makes decisions in the face of uncertainty about the future, investment decisions are always based on predictions about the future (often the distant future). Furthermore, investment decisions frequently require judgmental estimates about future events. It is this lack of certainty about the future that makes capital budgeting decisions one of the most challenging tasks. As the problems of investment decisions have become more complex and are characterized by a high degree of uncertainty, increasingly sophisticated analytical methods have become available for helping the businessmen in analyzing investment decisions.

The capital budgeting process embraces a rather broad and diverse class of activities associated with the allocation of capital resources to meet the demands for these resources. Essentially, these activities include the administration and organization of a capital expenditure program, the development of new investment opportunities, the estimation of future cash flows of investment proposals, the review of the investment progarm. However, this study is concerned primarily with the analytical techniques utilized during the decision-making phase of the capital budgeting process (that is, techniques to assist in capital rationing).

Capital rationing can defined as a situation in which a firm (or an organization) does not have and cannot obtain enough capital to make all of the investments that are available to it. The paramount problem that confronts the decision maker is to determine how the available capital should be allocated to the investment proposals that are competing for these funds.

1.1 Statement of the Problem

1.1.1 Decision Process

One of the important considerations in the evaluation of investment opportunities is the type of decision procedure which a firm follows in a capital expenditure program. In general, two classes of decision procedures can be identified. One of them is a decision process in which the decision maker makes a single decision about the investments to undertake within the entire time horizon. The other class represents a multi-stage decision process in which the firm makes investment decisions on a regular periodic basis throughout the time horizon.

Of particular interest is the decision process where a firm makes capital investment decisions on a regular periodic basis. Each decision period, the decision maker examines a set of investment proposals submitted for consideration during that period and then makes the investment at the end of the period. Once the allocation decision is made according to some decision criterion, the funds budgeted for that period are invested in those selected investment proposals.

1.1.2 Uncertainty about Future Investment Opportunities

When investment decisions are made on a regular periodic basis, one of the important considerations is the amount of information the decision maker can obtain about the future. One view of this problem is that the decision maker at the time of decision has complete knowledge about the investment opportunities that are to be selected for implementation in both the present and future. Another view of this problem is that the decision maker does not have any knowledge concerning future investment opportunities.

The assumption that a decision maker in most real world situations will have either complete information or no information about the future seems quite improbable. This study utilizes an approach which describes an investment framework that allows the decision maker some expectation as to future investment opportunities without requiring specific knowledge about particular investment proposals. This view describes some middle ground concerning the availability of information regarding the outcomes of future investments.

1.1.3 Uncertainty about Future Cash Flows

In view of the fact that most investment situations are stochastic in nature, no complete information regarding future cash flows of the investment proposals is assumed. At each decision period, cash flows commonly are projected at the time the investment is first proposed, and at least implicitly, the future cash flows are considered to be subject to probabilistic deviation from their expected values. That is, while initial outlays in a given project are known with certainty, the future cash flows are only estimates that can be described by known probability distributions.

1.1.4 Uncertainty Resolution about the Cash Flows through Time

One of the primary interests in this research is to recognize the fact that uncertainty about project cash flows over time is resolved as one moves through the decision-making process [81, 95, 104]. Cash flows are projected at the time the investment is first proposed, and as time passes and the future cash flows are realized, the project's overall prospects come into sharper focus. The process of moving from greater uncertainty toward less uncertainty is referred to as uncertainty resolution. The uncertainty arising from the yet to be realized cash flow outcomes for a proposed investment would be completely resolved at the end of the investment period. Thus, from the point of view of uncertainty resolution, the information of importance is how uncertainty about the project's future changes between project initiation and termination.

The significance of the manner in which uncertainty is expected to be resolved over time in the capital rationing context is that it provides the decision maker with considerable insight about changes in the level of uncertainty as the project estimates become realized. In a sequential decision-making process where future budgets are influenced by current decisions, this information about the proposal's level of uncertainty becomes extremely valuable to the decision maker. Therefore, the idea is, " How can this time-phased information regarding uncertainty about the cash flows be utilized in the investment decision-making process ? "

1.2 Objectives of the Research

The primary purpose of this research is to develop a methodology for measuring the resolution of uncertainty in quantitative fashion and to incorporate this information into the improvement of capital allocation decision problem. Another purpose of this research is to develop an understanding of a regular periodic investment decision process where the decision maker has neither complete information regarding future investment opportunities nor complete information regardidng the cash flows of the investment proposals. Of particular concern is the development of a decision criterion to be used in making investment decisions where current decisions are influenced by future investment opportunities and future budgets will be influenced by current decisions. These objectives are accomplished in three ways.

1. A principle concerning the measure of the resolution of uncertainty resolution is developed and incorporated into a decision criterion which can be used in making investment decisions on a regular periodic basis. The criterion is the <u>Project Balance Criterion</u> developed by the author.
2. By applying the Project Balance Criterion and three other decision criteria to identical groups of projects through computer simulation, the effectiveness of these criteria is compared. Computer simulation is utilized because it was necessary to investigate a large class of investment settings which were not manageable by available analytical techniques.
3. Data describing important features of this type of periodic decision process are generated and recorded to provide a better understanding of certain characteristics of the model's performance.

1.3 Plan of Study

Chapter II provides a review of the literature related to the issues raised in the various areas of the research problem. The review of the literature indicates that normative analysis of the problem of uncertainty resolution requires a time-dependent measure of investment worth which reflects the magnitude of the cash flow pattern over time.

Chapter III investigates time-dependent measures of investment worth which provide a basis for measuring uncertainty resolution. Two different measures of investment worth which reflect the shape of the cash flow pattern over time are examined, and the usefulness of these measures as a basis for quantifying uncertainty resolution is also discussed. Then, project balance concept is precisely defined.

Chapter IV discusses time-dependent measures of uncertainty resolution. A measure of uncertainty resolution based on the concept of project balance incorporated with the expected gain confidence limit criterion is formally proposed.

Chapter V incorporates the concept of uncertainty resolution which was developed in Chapter IV in a systematic manner into a framework for investment decisions. To utilize information about the resolution of uncertainty over time, a decision criterion called the Project Balance (PB) criterion is developed as a method to assess the effect of investment projects that have probabilistic outcomes. The precise description of the other criteria to be compared with the PB criterion is presented. The assumptions that underlie these criteria are defined here, and they are related to the actual conditions experienced by those making investment decisions.

Chapter VI describes the features and assumptions of the simulation model which is used to test the effectiveness of these criteria in a regular periodic investment decision process. The input parameters, the shapes of the probability distributions used, the method of generating probability trees, the starting conditions and other elements of the simulation are presented.

Chapter VII presents the simulation results and the analyses of the data regarding the objectives of this study. Three types of investment situations are described and their specific investment parameters are defined. Based on these investment settings, the effectiveness of the PB criterion is compared to the other tested decision criteria. To examine the effects of critical input parameters on the performance of each decision criterion, the sensitivity of the specific input parameters is analyzed.

Chapter VIII contains conclusions and recommendations for further research. Detailed mathematical derivations for certain problems in Chapter III and Chapter IV are presented in the Appendices.

CHAPTER II

REVIEW OF RELATED LITERATURE ON CAPITAL BUDGETING

In view of the significance of the investment decision-making process of the firm, extensive effort has been directed at the problem of capital rationing under risk. The ideas and results of this effort have been reported in the literature of a variety of disciplines (accounting, business, economics, financial management, operations research, and industrial engineering). In particular, the importance of considering risk in capital investment decisions has been stressed in the recent literature dealing with capital budgeting.

The review of the literature begins with a discussion of general capital budgeting problems. Depending upon the degree of information one can obtain about the future, capital budgeting decision models can be classified into two categories: deterministic models which assumes certainty, and nondeterministic midels which assume a probabilistic future. Of particular interest in this research is the area of nondeterministic models.

Distinction between risk and uncertainty has been made in this presentation. Decision-making under risk refers to the situation where the probability distributions describing the possible outcome of future events are known or assumed. The term uncertainty indicates that so little is known about the possible future outcomes that a reliable probability distribution cannot be described. This study assumes that the decision maker can describe probability distributions concerning future events, and therefore a quantitative statement about probabilistic outcomes can be made.

Since the primary interest of this study is in the area of capital rationing under risk, Section 2.2 examines two important areas in the literature (1) how the concept of risk is defined, and (2) how the concept of risk is recognized in the decision-making process. Tools to evaluate risk in an explicit manner are discussed in Section 2.2.2 and rationing criteria under risk are reviewed in Section 2.2.3.

In Section 2.3, the concept of uncertainty resolution is defined and a review of the literature on the subject is provided. In particular, this section explores the usefulness and limitations of the measures of uncertainty resolution currently available in the literature. Uncertainty resolution is of particular interest because time-phased information regarding the riskiness of projects can be useful in aiding a decision maker who has to make investment decisions on a regular periodic basis.

Section 2.4 considers the problem of the decision process. Two different decision processes can be investigated. One model requires that a firm makes decisions sequentially as investment proposals are presented for consideration. The other model describes a firm that makes investment decisions about batches of proposals at regular periodic intervals. One of the interests of this study is to develop an understanding of a decision process where a firm makes capital investment decisions on a regular periodic basis.

Finally, of particular interest is an investigation of a regular investment process where current decisions are influenced by expectations about future investment opportunities and future budgets will be influenced by current decisions. Therefore, when the firm makes investment decisions on a regular periodic basis with uncertain future investment opportunities, obtaining time-phased information about a project's remaining uncertainty is expected to be important. Since the concept of uncertainty resolution does provide this type of information, the use of this concept in a regular periodic decision process is presented in Section 2.4.3.

2.1 Capital Budgeting Problems

As defined in Chapter I, capital budgeting is a many-facet investment activity dealing with the effective utilization of resources. In general, the basic activities involved in many organizations can be divided into two categories: (1) administration of the capital budgeting program, and (2) capital rationing decision (or capital budgeting decision).

The first category embraces a variety of activities, all of which are essential to a successful capital budgeting program. These administrative activities include search for investment opportunities, estimation of cash flows, the forecasting of the availability and cost of funds for investment (financing decisions), project implementation and post audits. Obviously, these administrative activites constitute some of the elements most critical to the success of an investment program. They may also represents some of the most difficult activities to accomplish in the capital budgeting process.

The second category, capital rationing decision, is of primary interest in this study. Capital rationing deals with the problem of allocating limited resources among competing investment opportunities. In particular, when a decision maker is faced with a situation in which he must select among alternative investments under various constraints, he is faced with a capital rationing decision.

When one makes a capital rationing decision, one of the important considerations is the amount of knowledge one can obtain about the future. If the decision maker at the time of decision has complete information regarding the investment opportunities that are to be selected for implementation in both the present and future, this situation is referred to as deterministic capital rationing decision. On the other hand, if the decision maker has less than perfect information or no information at all regarding the investment opportunities in either present or future (or both), this situation could be described as stochastic, probabilistic or nondeterministic.

2.1.1 Deterministic Capital Rationing Decision

Numerous models have been appeared in the literature as rational approaches for dealing with deterministic capital rationing problems (Dean [20], Daver [19], Fleischer [26], and see also a comprehensive review in Bernhard [5]). Analysis for capital rationing decisions has become more sophisticated through the application of new mathematical tools. The most comprehensive treatment of the problem has been by Weingartner [101]. His basic formulation, which uses the Lorie-Savage problem [56] as a point of departure, has made widely known the usefulness of mathematical programming for allocating capital under various constraints.

When multi-period rationing decisions are considered, the effectiveness of using mathematical programming in capital rationing becomes more pronounced. In other words, use of mathematical programming permits the whole set of investment alternatives to be considered in both the present and future as a program. Complex interrelationships among investment projects can be stated and analyzed at one time as can the financial interrelationships imposed by capital rationing (see the basic horizon model formulated by Weingartner [101].

The principal difficulty of mathematical programming approaches to capital budgeting, however, is that they are based on the assumption that all future investment opportunities are known with certainty. The assumption that a decision maker in most real world situations will have complete information about the future seems quite unlikely. In view of the fact that most investment situations are stochastic in nature, the main interest of this research focuses on the nondeterministic capital rationing decision problems.

2.1.2 Nondeterministic Capital Rationing Decision

Since the future is rarely known with certainty, capital rationing decisions are normally based on predictions about the future. Variations in the outcomes of the future events (variations from the judgmental estimates) have been the primary concern to most decision makers in the evaluation of investment proposals (see Hicks [42], pp. 126-127 for classical reference on this subject).

Decision situations may be broken down into two categories depending upon the degree of difficulty in predicting the future: risk and uncertainty. The distinction between risk and uncertainty is that decisions under risk are referred to as those situations in which the probability distributions of future events are either known or assumed. For decisions under uncertainty, these probabilities are not known, or the decision maker is not willing to (or feels he is not able to) state quantitatively the probability associated with possible future events.

The primary interest of this research is to investigate capital rationing under risk. However, it is recognized that there may be many situations in which the decision maker may not be able to assign probabilities to future events so he must rely on a totally different approach for formally considering this lack of information (Luce and Raiffa [57], Fishburn [25] and Adelson [1]).

2.2 Capital Rationing Considering Risk

Accounting for the risk involved in alternative investment proposals is often one of the important considerations in the evaluation of these proposals. To gain insight into the research problem addressed in this work, a review of what has been occurring in both the applied and theoretical areas of capital rationing decisions considering risk is deemed worthwhile. Thus, this section examines two important areas in the literature: how the concept of risk is defined, and how the concept of risk is recognized in the decision-making process.

2.2.1 The Concept of Risk

The concept of risk most widely used in the literature is the variability of return, which is measured by variance (or standard deviation) [62, 93]. That is, the more an investment's return varies about its expected return, the larger is the investor's risk. When variance is used as a measure of risk, it implies that deviations below expected value are regarded the same as deviations above the expected value. Even though this measure has been criticized as too conservative since it regards all extreme returns as undesirable, whether positive or negative, variance is a popular measure of risk because of its familiarity and ease of computation [61].

As an alternative, semivariance, which is a similar approach to risk in problems of this type, has the advantage of focusing on reduction of losses, that is, variability in negative return [61]. When this measure is used in capital budgeting problems (for example, portfolio selection), it requires a full knowledge of the joint probability distribution about the projects' investment returns. This information is required for calculating the values of mean, variance, and semivariance of alternative investment portfolios. When the decision maker is faced with a set of alternatives and each project has a large number of outcomes (for a discrete case), the development of such a distribution is usually impractical.

As mentioned earlier, the use of variance as measure of risk treats with indifference potential outcomes above and below expected value. Thus, a modified version of this measure is proposed by Baumol [3], which is referred to as the expected gain confidence limit criterion (EGCL). This supplementary measure takes the form of $L = E - \delta\sigma$ where E and σ stand for the expected value and standard deviation of the net present value about the mean of an investment proposal, and δ is a parameter which represents the degree of risk aversion of the decision maker. Here L is said to be the critical point on which an investment decision should be based.

Another plausible measure of risk in capital budgeting literature would be the probability of loss criterion. This measure, along with some variants of it, has become known as the safty-first rule [83]. If risk is defined as the chance of loss, risk is measured by the area of a probability distribution which lies below the point of profitability. In other words, this is a measure which treats only unfavorable return as equivalent to "loss." This implies that the risk of a project increases if the likelifood of loss increases or if the magnitude of the possible loss increases. For example, a project with a large variation of profit may have no possibility whatever of loss. In fact, such a project would be viewed as risk free by those who use this criterion, however great the variability of potential outcomes. However, to compute the probability of loss or the expected loss, the complete knowledge of joint probability distributions of investment proposals is required. These distributions can be rather difficult to obtain [61].

Despite the many measures of risk discussed in the literature, it is observed that there is considerable disparity between the definition of risk and the measures of risk that businessmen are applying and those that the academicians recommend. A recent survey by Petty, Scott and Bird [73] indicates that businessmen viewed risk as being primarily concerned with the probability of not achieving a target return. Almost 40% of the the corporate executives interviewed described risk in this manner. In other words, management is more concerned with negative variation rather than total variation of possible investment outcomes. The second leading definition of risk in this survey relates to variation in returns, which is equivalent to variance as measure of risk. Their are, in general, consistent with those previous studies by Mao [60] and Klammer [52].

2.2.2 Methods for Quantifying Risk

Numerous approaches have been proposed for incorporating risk quantitatively in investment analyses. Most frequently, however, the methods recognized for analyzing the riskiness of a capital investment are referred to as the probability distributions approach and the utility approach. There are a few notable exceptions such as the risk-adjusted discount rate approach, the payback period approach and the variation of project life as a means for adjusting risk.

The two methods, the risk-adjusted discount rate and the payback period, have been discussed in the literature as an approach for considering risk in the evaluation of investment proposals [80, 96, 54, 69, 104]. The major difficulty associated with the risk-adjusted discount rate approach is in determining the approapriate discount rate for a particular investment [96,Ch.5]. This determination is likely to be somewhat arbitrary so that this approach fails to deal with such risk in an explicit and accurate fashion. On the other hand, the serious drawback of the payback method as an approach for considering risk is that it places no time value on the stream of revenues generated after the project recovers its initial investment. However, it is recognized that in practice there is still a strong tendancy to associate the riskiness of an investment with the number of years to recover the original investment [73, 104].

In capital budgeting practice, a number of firms adjust for risk by reducing the life of the project the greater its perceived risk [9].

Van Horne [97] analyzes the variation of project life as a means for risk adjustment, and identifies the various biases inherent in its use.

2.2.2.1 Risk Represented by Certainty-Equivalent. The certainty-equivalent approach has been discussed by Robichek and Myers [80] as an alternative approach to the risk-adjusted discount rate approach. With this method distributions of possible cash flow outcomes are specified period by period and a certainty equivalent is substituted for each of the distributions. Then, the modified cash flows are discounted back to present value at a risk free interest rate. In particular, the adjustment in the estimated cash flows is carried out by mangement's utility preferences with respect to risk.

Like the risk-adjusted discount rate approach, the certainty-equivalent approach presents practical problems of implementation. With the certainty-equivalent approach, the adjustment in a stream of cash flows is to be determined by specifying the certainty-equivalent coefficients period by period. The most difficult problems are in specifying the appropriate degree of risk in terms of these coefficients for an investment opportunity and in being consistent in these specifications from project to project [96]

2.2.2.2 Risk Represented by Probability Distributions. A more sophisticated approach which considers the probability distributions of cash flows over time in the evaluation of risky investments has received wide attention in the literature. The idea is to present management with pertinent information about the expected value of return and the dispersion of the probability distribution of possible returns. In the literature two methods have been utilized to define the measure of investment worth as a random variable. The two methods are the analytical method and the Monte Carlo method.

An analytical method developed by Hillier [43, 44] and implemented by Wagle [100] shows how the exact mean and variance of the probability distribution of the present value can be derived. With Hillier's results as a point of departure, many writers have extended the ideas presented in Hillier's previous paper [43], generalizing to the various cases of investment situatuions (Fairley and Jacoby [23], Kabak and Owen [50], Canada and Wadsworth [15], Perrakis and Henin [70], Perrakis and Sahin [71], Mantell [59], and Young and Contreras [105]).

Hertz [39] is one of the first to suggest the use of Monte Carlo simulation to accommodate the riskiness of an investment. The popularity of using this particular technique to obtain the expected mean and dispersion about the expected return for an investment lies in the fact that for complex situations, the mathematical calculation of the required statistics is impossible or extremely difficult. Indeed, most of the methods used today employ this approach since it provides not only the expected net present value, but also the shape of the distribution of the criterion function [11, 40]. Certain limitations to this technique have been discussed by Hillier [45].

2.2.2.3 Risk Represented by Utility Functions. The approaches discussed in the previous section have all been based upon the use of probabilistic monetary values. The utility theory approach incorporates the utility preferences about risk into the investment decision by use of utility function [77,Ch.5]. Even though this approach has been discussed extensively in the literature [31, 44, 67] as rational approach to incorporate risk in investment analysis, the approach is infrequently used in practice [52,73]. The most difficult requirement of this approach is to derive and specify the utility function numerically for various investment situations.

In summary, of all the methods for dealing with risk, the direct use of probability distributions appears to be the most feasible and practical. Therefore, the following section reviews the literature in the area of capital rationing criteria with probabilistic considerations.

2.2.3 Capital Rationing Criteria with Probabilistic Considerations

Most often, investment proposals have been analyzed by using net present value as a criterion function, even though the choice of criterion for optimization is rather difficult. A firm may be primarily interested in profitability in the evaluation of investment proposals, but it may entertain a host of other nonfinancial considerations [29].

When nonfinancial considerations such as labor relations, its public image to stockholders, share of market, safety of employees or the the public are factored into the investment decision analysis, multi-attribute utility functions' approach has been considered. These utility functions summarize all the relevant information about trade-offs among the many investment factors [77,Ch.9]. With recent advances in multiple objective programming, Ignizio [48] discusses the capital budgeting problem having multiple objectives in a mathematical programming context. Although it is recognized that there is substantial evidence that suggests business firms have nonmonetary objectives and pursue nonmonetary goals [29], these considerations are not investigated in this study.

Numerous decision criteria have been appeared in the literature for evaluating the profitability of risky investment proposals with budget constraints [2, 12, 17, 27, 30, 36, 37, 62, 65, 72, 89, 102]. When the objective of the decision maker can be viewed as the selection of the set of projects which maximizes some criterion function, the form of the criterion function which leads to the desired conclusion is important. Two different views on the form of the criterion function are discussed in the literature. One view is that the function is linear in the outcomes, while the other view assumes a nonlinear relationship [102].

2.2.3.1 Expected Value Maximization. For this criterion, the decision maker is assumed to be risk indifferent such that his problem is to select the feasible solution vector having the largest expected net present value without violating the budget constraint. Assuming that the decision maker has defined meaningful random variables and that he knows the shapes and parameters of their distributions, he can compute his expected payoff without considering correlations among the projects. Even if the outcomes of investments are jointly distributed, nothing new is introduced under the expected value maximization criterion. This type of model has been discussed by Weingartner [102], Peterson and Laughhunn [72].

Weingartner [102] also discusses the second-order effects for interrelated projects if the conditional distribution of outcomes of a particular project, given that another is undertaken, is different from the unconditional distribution. This type of model which has the objective of expected value maximization is formulated by quadratic integer programming and solved by Reiter's method [78].

A different model which also employs expected value maximization is the chance-constrained model of Byrne, Charnes, Cooper and Kortanek [13]. In this model, a probabilistic payback constraint is imposed over all investments simultaneously so the specific objective is to select those projects which satisfy the constraints while maximizaing expected net present value. This type of model has been criticized for its lack of realism [60].

2.2.3.2 Mean-Variance Criterion. Markowitz [62] has proposed an approach consisting of successively minimizing a portfolio's variance for each of a number of expected values or expected returns. Markowitz's approach is frequently referred to as the portforlio approach.

When the portfolio approach is applied, the model is generally an adaptation of the Markowitz protfolio selection model with 0-1 conditions imposed on the decision variables to reflect project indivisibility [102]. The model requires that the decision maker specify the rate of trade-off between reduction in expected value for reduction in variance.

The Markowitz mean-variance criterion assumes that variance of return as the measure of risk. However, different models which utilize the standard deviation are formulated by Peterson and Laughhunn [72]. When the expected gain condifence limit is used as the measure of risk, a model which also employs the portfolio approach is discussed in [72]. If risk is defined as the chance of loss, Roy's safety-first rule model would be another adaptation of the Markowitz mean-variance criterion [83]. On the other hand, Mao [61] formulates the portfolio selection model by utilizing the semivariance as a measure of risk.

Markowitz's portfolio selection model requires that a problem be solved with the full variance-covariance matrix, which is quite time consuming, primarily due to the matrix inversion required for computing the critical line. Thus, Sharpe [86] offers an alternate computational scheme, the Diagonal Model (or Index Model), which assumes that the comovement of securities depends only on a single market index. Some accuracy is sacrificed, but the reduction in complexity is significant.

In conjunction with the mean-variance criterion, the stochastic dominance criterion also has appeared in the literature [34]. Stochastic dominance selection rules utilize every bit of information in the probility distribution rather than simply focus on the probability distribution's first two moments. As a result, stochastic dominance selection rules occasionally can yield portfolios which maximize expected utility and are not Markowitz efficient portfolios [35]. Although stochastic dominance selection rules are logically superior to other simpler decision criteria, their practical value is dubious because they require knowledge of every point on the probability distribution rather than merely the first two moments.

2.2.3.3 Expected Utility Maximization. When the objective of the decision maker is to select the set of proposals which maximizes expected utility, but the utility function is not linear in the outcomes, then the nonlinear utility criterion function gives rise to a problem not present when simple expected value of outcomes maximization serves as the criterion [102]. Utility is defined for the entire portfolio of projects, including both the projects being considered and those already and in the process of being carried out. Weingartner [102] develops some analytical concepts in handling these types of problems.

Apart from Weingartner's formulation, Hillier [45] provides a utilitarian approach to the problem of selecting the best combination of the investments under consideration by deriving the probability distribution of the present value for each feasible combination of investments. The objective is to seek the feasible combination that maximizes the expected value of a utility function.

In summary, for the models discussed above, the emphasis has been on anlyzing the effects of risk of future cash flows which result from an investment on the economic desirability of that investment. The most difficult problem in evaluating risky investment proposals with budget constraints is the choice of the criterion function for maximization. Determination of the objective function to be maximized is dependent upon the decision maker's attitude toward risk preferences.

2.3 Uncertainty Resolution in Capital Budgeting

Relatively little attention has been focused on the concept of uncertainty resolution as a way to be more efficient in the selection of investments. In particular, none of the models discussed in the previous sections deal with the concept of uncertainty resolution in an explicit manner. In fact, Robichek and Myers [81], apparently the first to use the term "resolution of uncertainty," discuss the potential importance of the concept for an investor selecting a portfolio among securities which have different uncertainty resolution patterns. These authors deal only in part with the question of ". . . the manner in which uncertainty is expected to be resolved over time ([81], pp. 224-227)" from the viewpoint of equilibrium in financial markets.

In view of the significance of the concept of the resolution of uncertainty in this research, it is worthwhile to illustrate what is meant by uncertainty resolution with a simple numerical example.

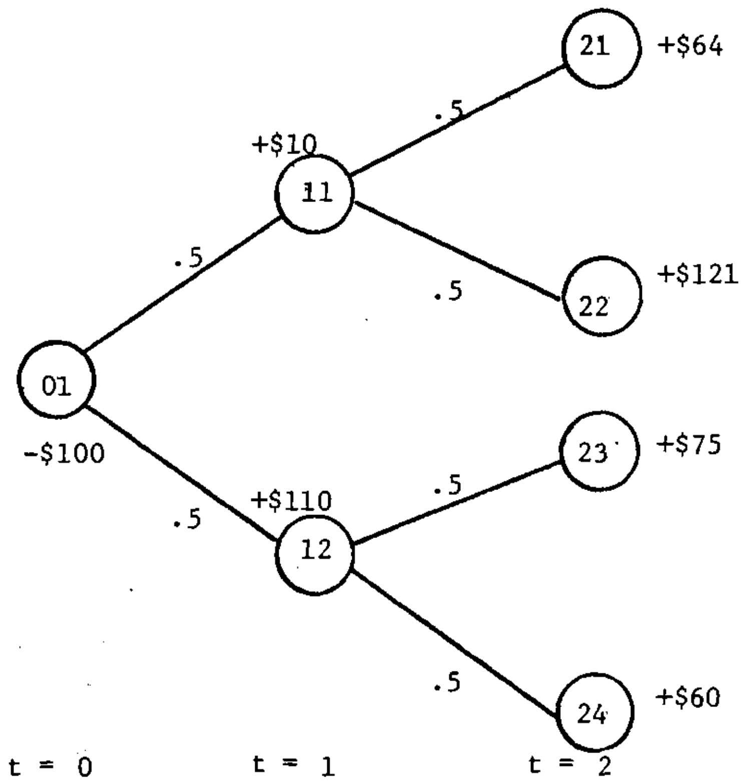


Figure 2-1. Illustration of Uncertainty Resolution

In the previous example in Figure 2-1, the project requires an initial outlay of $100. Between Period 0 and Period 1, there is a .50 probability of moving to Node 11 and realizing a cash inflow of $10, versus a .50 probability of moving to Node 12 and obtaining a cash inflow of $110. Futhermore, in each case the probabilities and cash flows associated with Period 2 are as stated in Figure 2-1.

When period t = 1 occurs, either Node 11 or Node 12 will have been realized, each with probability of .50, and some of the initial uncertainty concerning the future return from the project will have been eliminated. Specifically, the investor will have learned by time t = 1 whether the $100 associated with Node 11 will be obtained or not, and considering all the possible cash flows in Figure 2-1, most people would agree that the major uncertainty about the project's desirability involves the $100 cash flow. Thus, as time passes and the future cash flows are realized, the project's overall prospects come into sharper focus. <u>The process of moving from greater uncertainty toward less uncertainty is referred to as uncertainty resolution</u>.

There are three major approaches in the literature which deal with the concept of uncertainty resolution in explicit manner. The three approaches are the payback period method, the coefficient of variation method, and the certainty-equivalent method.

2.3.1 Payback Period as a Measure of Uncertainty Resolution

As a measure of uncertainty resolution, Weingartner [104] discusses the payback concept as an approximate measure of uncertainty resolution. He argues that the rate at which the uncertainty developing around the outcome is expected to be resolved may be measured, at least crudely, by the relation between the series of cumulated expected cash inflows and the amount of the investment. Thus, he emphasizes that one virtue of the payback method of capital budgeting analysis is that it is a measure of the rate at which uncertainty is expected to be resolved. However, he neither argues that one should use the payback period method for decision-making purposes, nor shows how one can translate this payback concept into the sequential capital budgeting decision problems.

The serious drawback of the payback method as a basis for measuring uncertainty resolution is that it places no time value on the stream of revenues generated after the project recovers its initial investment. Another difficulty associated with the use of the payback method is that it is rather difficult to find a meaningful index which represents the rate of the resolution of uncertainty through time, when a proposal's cash flows are expressed in terms of a probability tree. One may compute the expected payback period and variability about the expectation for a proposal shown in Figure 2-1 [63]. However, the interpretation of the the statistics as to the rate of uncertainty resolution over time is rather vague.

2.3.2 Coefficient of Variation as a Statistical Measure of Uncertainty Resolution

Van Horne [95] proposes a methodology for analyzing how uncertainty is resolved over time in the case of new product investment. His approach consists of dividing the square root of the weighted average of possible variances (about the relevant conditional means) at any time period t by the total expected discounted terminal value of the investment.

Van Horne argues that if an ivestor wishes to maximize net present value subject to maintaining its risk complexion, this pattern of uncertainty resolution discloses what types of projects the investor will need to generate. That is, if uncertainty is expected to be resolved very quickly for the existing portfolios and, for some reason, the opportunities the firm expects to arise are "less" risky in the near future, then management would be able to consider relatively risky projects at the current decision point in its attempt to maximize net present value subject to maintaining its risk complexion. This is because the maintenance of an approximate constant risk posture need not involve a constant level of uncertainty over time from the existing portfolio of investments.

2.3.2.1 Bierman and Hausman's Critique of Van Horne's Measure. In conjunction with Van Horne's development, Bierman and Hausman [8] explore the usefulness and limitations of the concept of uncertainty resolution in the evaluation of both single risky investments and in portfolios of risky investments. Their major conclusion is that Van Horne's measure of uncertainty resolution for a single investment does not provide complete information and, in situations in which all future investment opportunities are known, a measure of uncertainty resolution for individual investments is not needed for systematic analysis of the portfolio type of investment problem. On the other hand, for situations in which future investment opportunities are not known with certainty, the concept and Van Horne's index for portfolio resolution of uncertainty do seem to have usefulness in aiding the firm in its attempts to maintain a given risk profile. However, Bierman and Hausman did not attempt to develop any alternative measure of uncertainty resolution.

2.3.2.2 Difficulty Associated with Van Horne's Measure. Apart from Bierman and Hausman's criticism, the usefulness of Van Horne's approach as a measure of uncertainty resolution is open to question on at least one point. As stated earlier, this methodology is equivalent to dividing the square root of the weighted average of possible variances at any time period t by the total of the investment. Since this computation is based on solely on the magnitude of terminal value, the index fails to consider the cash flow pattern of the probability tree. To demonstrate what is meant by failure to fully consider the cash flow pattern of the probability tree, consider Projects Z1 and Z2 shown in Figure 2-2.

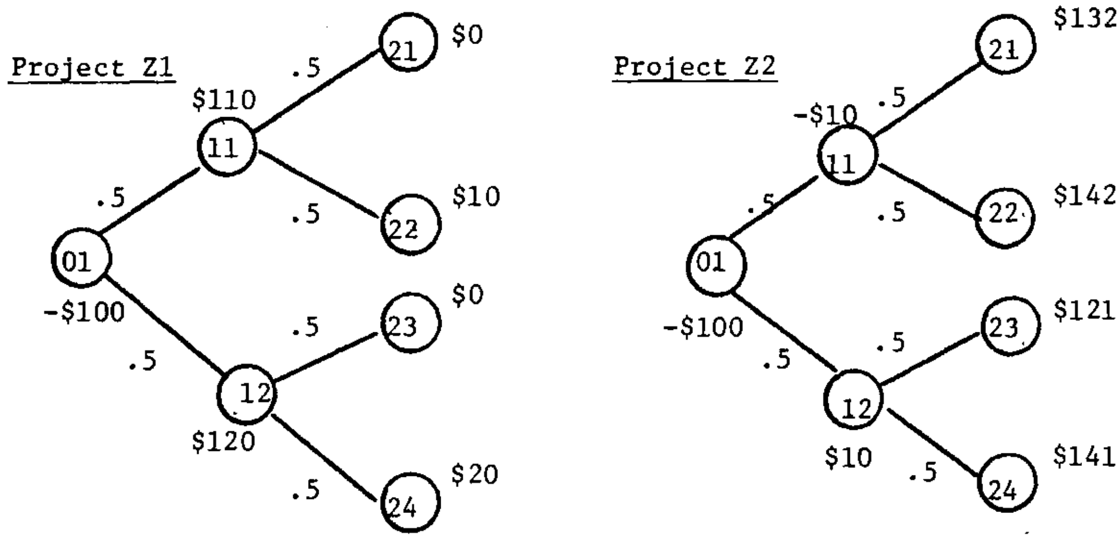


Figure 2-2. Probability Trees Associated with Project Z1 and Project Z2

As shown by Bierman and Hausman [8, pp. B-656-658], Figure 2-2 can be transformed into a probability tree with net future equivalent value (or net discounted present value) occurring only at the branch tips in the final period as shown in Figure 2-3. The actual resolution of uncerainty is identical in the two probability trees of Projects Z1 and Z2 shown in Figures 2-2 and 2-3. Since the two probability trees in Figure 2-3 have identical terminal values with identical conditional probabilities, Van Horne's measure will indicate that both projects are equivalent in terms of the way of resolution of uncertainty over time.

A close examination of the probability trees reveals that Project Z1 recovers its initial investment relatively faster than Project Z2 does no matter which branch is realized. In general, if the cash flows

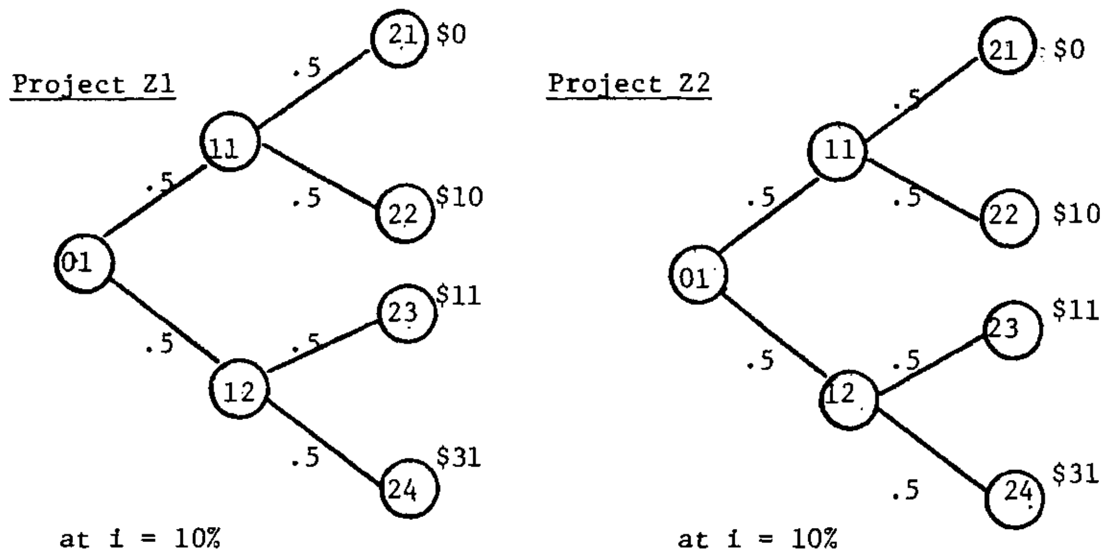


Figure 2-3. Alternate Probability Trees for Projects Z1 and Z2

are highly variable, then the uncertainty of a project will be reduced if it pays for itself quickly. In other words, the investor will not have to wait for events occurring many years in the future before he knows whether he will recover the outlays that have been made. In that sense, it can be said that Project Z1 quickly resolves uncertainty associated with its future receipts as compared to Project Z2. From the standpoint of an investor, this information should be extremely valuable because a project's ability to return its investment would be of prime concern to the investor faced with the uncertain investment opportunities of the future. As demonstrated in the above numerical example, Van Horne's measure does not provide complete information, as it is solely based upon the magnitude of terminal profitability.

Even with certainty, it is not difficult to show that the present worth measure (i.e., based on terminal profitability) is not a sensitive time-dependent measure of investment worth which reflects changes in the cash flow pattern as a function of the project's life. To illustrate what is meant by an insensitive time-dependent measure of investment worth, consider the following two investment proposals in Figure 2-4.

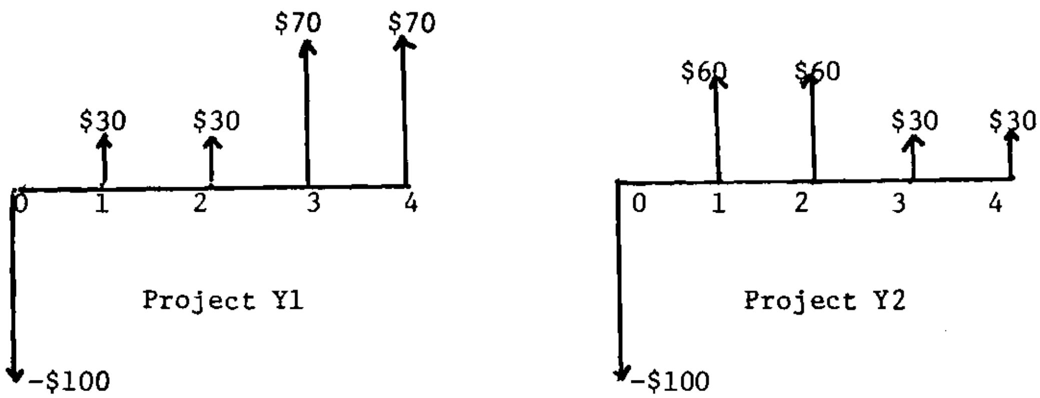


Figure 2-4. Cash Flow Diagrams for Proposal Y1 and Proposal Y2

At an interest rate of 15.5%, the net present worth of both investments would be $33.23. In other words, under certainty, Proposals Y1 and Y2 are equivalent in terms of economic desirability. This is because the present value represents the aggregate terminal profitability measured at the present point in time. However, this measure fails to fully consider the shape of the cash flow pattern over the project life.

In terms of the project's ability to return its investment, Proposal Y2 recovers its initial investment faster (two years) than Proposal Y1 does (three years). When the decision maker anticipates another investment opportunity within the lives of the projects, he may not be indifferent in selecting a project even though the present value of each project is identical with certainty.

Under conditions of uncertainty, the shape of the cash flow pattern will be of importance because it provides information about the rate at which the uncertainty developing around the outcome is expected to be resolved. If a project has a quick expected recovery of its initial investment, the decision maker is left only with the problem of whether the useful life of the project beyond the full recovery period (payback period) is profitable enough to make it attractive.

2.3.3 Certainty-Equivalent Approach for Considering Uncertainty Resolution

Apart from the coefficient of variation approach, Percival and Westerfield [69], Cozzolino and Zanhner [18] develop a measure of uncertainty resolution based on utility theory. Basically their approach uses a consumption basis to create a preference structure for extended-time decision problems under uncertainty with reference to individual utility functions (however, those authors did not apply their methods to capital rationing problems).

The major difficulty associated with the certainty-equivalent approach is in the development of an appropriate utility function to identify the time preference of consumption. In particular, an indvidual's time preference for future consumption depends largely on what investment opportunities this individual would have in the future. However, the occurrence, timing, and characteristics of future investment opportunities are rather difficult to predict with certainty in most real investment situations. Under these circumstances, it is rather difficult to build a rigorous utility function which describes the time preference of consumption in a meaningful way.

2.4 Multi-Stage Capital Budgeting Decisions

As outlined in Chapter I, the primary purpose of this research is to develop an understanding of a decision process where a firm makes capital investment decisions on a regular periodic basis. More specifically, for each period, the decision maker examines a set of investment proposals submitted for consideration during that period and then makes the investment at the of the period. Of particular interest is the study of a regular investment process where current decisions are influenced by expectations about future investment opportunities and future budgets will be influenced by current decisions.

When a firm makes decisions sequentially as investment proposals are presented for consideration, the apparent economic advantage is that there will be no needless delay in the implementation of highly productive investments. On the other hand, when the firm makes decisions with regular periodic intervals on batches of proposals, the apparant economic advantage is that the decision maker can be more discriminating when all the proposals are competing simultaneously. Para-Vasquez and Oakford [68] compare the relative effectiveness of these two procedures for simulated investment environments. They conclude that there are statistical differences between the two decision procedures and the firms should investigate the periodic (annual) decision-making procedure as an alternative to sequential decision-making.

2.4.1 Periodic Decisions with Imperfect Knowledge of Future Investment Opportunities

When the firm makes investment decisions on a regular periodic basis, the amount of knowledge the decision maker can obtain about the future will be one of the important considerations in the capital rationing decisions. One view of this problem is that the decision maker at the time of decision has complete knowledge about the investment opportunities that are to be selected for implementation in both the present and the future. A more common and practical view is that the decision maker has complete information only about those investment opportunities at the time of decision (that is, the future investment opportunities are uncertain).

2.4.1.1 Stochastic Programming Approach. The problem of allocating a fixed set of time-phased investment decisions among competing investment proposals can be formulated by Weingartner's basic horzon model [101], if there is no uncertainty associated with the cash flows of each investment proposals. However, most investment situations are stochastic in nature and few project selection decisions are final ones based upon complete information.

When a proposal involves a number of continuation decisions during the evolution of the project (that is, a proposal that can be described by a decision tree), Lockett and Gear [55] present a method of formulating the capital budgeting problem which allows for risk and the multi-stage nature of the decision process. The approach utilizes a decision-tree method of representing individual projects and a form of stochastic integer programming to optimize the selection of a subset of projects.

In fact, their approach is an extension of Weingartner's mathematical programming formulation and allows chains of events to be represented with stochastic decision trees (see Hespos and Strassman [41] for the concept of the stochastic decision tree and Byrne, Charnes, Cooper, and Kortanek [14] for the formulation of a stochastic programming model).

The difficulty associated with the stochastic programming approach lies in the fact that as the problem size increases, the number of integer variables in the model becomes large and the solution cannot be obtained through currently available integer programming codes. For complex situations, as the authors admitted, resort to some form of simulation ultimately may be necessary.

2.4.1.2 Salazar and Sen's Model. When investment proposals are described by probabilistic cash flows, but the consequences of the future investment opportunities are known, Salazar and Sen [84] propose the use of stochastic linear programming to determine the expected return and variability of a combination of risky investments. They begin with Weingartner's horizon model for capital rationing, where the objective function is to maximize the future worth at some terminal date, subject to budget constraints.

Two types of risk are introduced in Salazar and Sen's model. The first involves possible changes in economic and competitive factors likely to affect cash flows. The second type of risk involves the variability in cash flows for given economic and competitive factors. With this probabilistic information, they propose simulating a series of future events by the use of Monte Carlo technique, then apply Weingartner's horizon model to the set of data obtained from the simulation. By repeated simulation of the future events associated with specific portfolios and horizon values obtained from the simulated set of portfolios, one can obtain the probability distribution of horizon values for each of the portfolios. By computing the expected values and standard deviations of the probability distributions, a risk-return chart is generated. Their approach is unique in generating a risk-return chart through linear programming and simulation. Nevertheless, some difficulty in the model is associated with the expression of the functional relationship between project outcomes and economic and competitive factors.

2.4.1.3 Thuesen's Model. When the consequences of the future investment opportunities are not known with certainty, none of those approaches described in the previous sections can be used directly. In particular, when the occurrence, timing, and characteristics of future investment opportunities are difficult to predict with certainty, one cannot build a rigorous model of this situation. In fact, it would be the case which fits the situation most businessmen face [8].

Very little work has been done on the model of this type of investment situation. Thuesen's doctoral dissertation [90] (also see Oakford and Thuesen [66]), which used simulation techniques, is a notable exception. The objective of Thuesen's model is to compare the performance of the Maximum Prospective Criterion [66] with the other decision criteria in a decision process where a firms makes capital investment decisions on a regular, periodic basis.

It is true that most of the literature discussing the differences between procedures is confined to the single stage decision (that is, single decision about the investments to undertake within the entire time horizon) and uses analytical techniques for making the comparison with a small number of proposals (see a few exceptions in Sundem [89], Para-Vasquez and Oakford [68]). Thus, Thuesen's simulation model provides a practical method for obtaining the large number of proposals that would be needed to compare the relative effectiveness of selected decision procedures on a regular, periodic basis.

In simulating investment situations, one of Thuesen's basic assumptions is that the decision maker lacks complete knowledge about future investment opportunities, but at each decision period, the decision maker has complete information about the characteristics of proposals considered at that decision time. This assumption, in fact, appears to be too restrictive in the sense that in most actual decision situations will seldom complete information about the proposals available.

2.4.2 Incorporation of the Concept of Uncertainty Resolution in a Periodic Decision Process

As discussed in Section 2.3, for situations in which future investment opportunities cannot be precisely known, the concept of uncertainty resolution is useful. In particular, Bierman and Hausman [8] explain how the information provided from uncertainty resolution of investment proposals can be helpful as follows:

> Such information (refers to uncertainty resolution) could be of use to a manager who has a basic feeling for the general manner in which new investment opportunities may occur for his particular company as time passes. Even though he is not willing to state precisely the probabilistic mechanism for generation of investment opportunities over time, it seems reasonable to presume that he has some information about this process and can therefore use portfolio uncertainty resolution information in his decision-making.

The potential importance of the concept of uncertainty resolution has been addressed by several authors as seen in Section 2.3. However, nowhere in the literature is there a successful application of the idea to a regular periodic decision process where the decision maker lacks full knowledge of his future as well as present investment opportunities. Therefore, it is the purpose of this research to investigate such a situation.

CHAPTER III

TIME-DEPENDENT MEASURES OF INVESTMENT WORTH

As outlined in Chapter II, this research considers the resolution of uncertainty through time for investments having uncertain future outcomes. Normative analysis of the problem of uncertainty resolution raises two basic questions: how to quantify information regarding uncertainty resolution and how to utilize the information in the improvement of capital budgeting decision problems. First of all, it is important to determine the basis on which uncertainty resolution will be measured. The appropriate basis of measuring uncertainty resolution should be the one which provides information about the changes in cash flows over time.

This chapter investigates time-dependent measures of investment worth which provide bases for measuring uncertainty resolution in the subsequent two chapters. Two different time-dependent measures of investment worth are discussed: the unrecovered balance method and the project balance method as a basis for measuring uncertainty resolution. Section 3.1 discusses the concept of the unrecovered balance method, its uniqueness, and the limitations of using the unrecovered balance method as a basis for measuring uncertainty resolution. Section 3.2 introduces the concept of project balance as time-dependent measure of investment worth in lieu of the unrecovered balance method.

3.1 Measures of Investment Worth

In Chapter II, it was observed that the traditional measures of investment worth do not provide complete information regarding the shape of the cash flow pattern. The shape of the cash flow pattern is important because it conveys information regarding the internal generation of funds from investments for the future.

When investment decisions are made on a sequential, periodic basis with funds generated internally, the manner in which funds become available in the future determines the flexibility of the firm's investment activities. Under conditions of uncertainty, the shape of a cash flow pattern will be of great importance because it provides information about the rate at which the uncertainty associated with the outcome is expected to be resolved. Obtaining information about the project's expectations is extremely valuable to the firm for making subsequent decisions, because the firm can evaluate the advantages of current consumption with respect to investing in future alternatives which may become available [104].

In this section, two different measures of investment worth which reflect the shape of the cash flow pattern over time are examined. These two time-dependent measures of investment worth are the unrecovered balance method and the project balance method. The usefulness of these measures as a basis for quantifying uncertainty resolution also is discussed.

3.1.1 Unrecovered Balance as a Time-Dependent Measure of Investment Worth

3.1.1.1 Unrecovered Balance--Basic Definition. One of the reasons for the use of the rate of return (ROR) in business is the fact that it conveys information about the relative desirability of alternative investments. In the economic sense, the rate of return represents the percentage or rate of interest earned on the unrecovered balance of an investment. Conversely, it means that exactly the original capital plus all accumulated interest has been returned at the end of the project. Therefore, the unrecovered balance can be viewed as the portion of the investment that remains to be recovered after interest payments and receipts have been added and deducted, respectively, up to the point in time being considered [92, Chap. 6].

The unrecovered balance at each point in time can be found from the following recursive equation,

$$U_t = U_{t-1}(1+i^*) + F_t \tag{3-1}$$

where F_t = The payment received at the end of period t

i^* = The interest rate earned on the unrecovered balance during period t, i.e., rate or return (ROR)

U_o = The initial amount of project investment

To illustrate the fundamental meaning of unrecovered balance for a project, consider the following two investment projects shown in Figure 3.1. Each investment requires the same amount of outlay, and the corresponding rate of return (ROR) for each investment project is computed at 28.5% for Project B1 and 34% for Project B2, respectively. By examining the cash flow patterns of both projects, it appears that Project B2 recovers its initial investment at a faster rate than does Project B1. The unrecovered balances related to each investment project can be found from Equation 3-1 and summarized in Table 3-1.

When the unrecovered balances are plotted as a function of time,

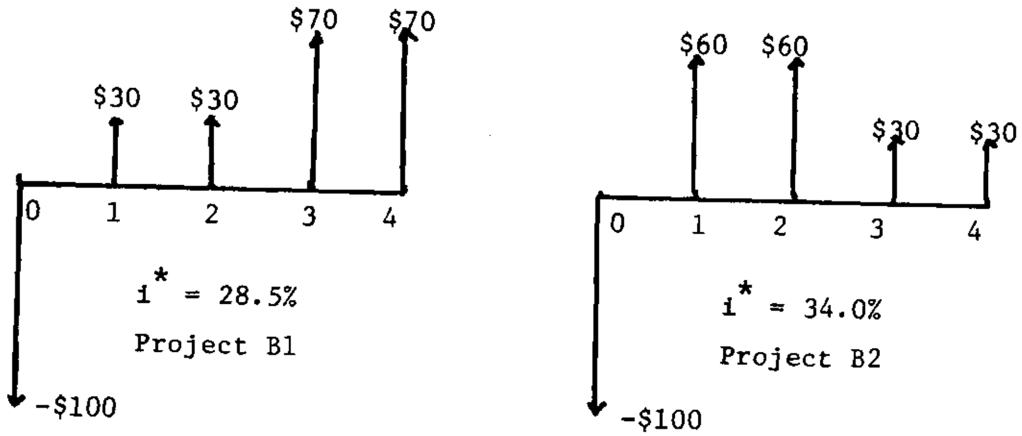


Figure 3-1. Cash Flow Diagrams for Projects B1 and B2

Table 3-1. Unrecovered Balances as a Function of Time

Project B1

Time (t)	0	1	2	3	4
Amount Owed (U_{t-1}) ($)	-100	-100	-98.5	-96.57	-54.38
Interest Owed ($U_{t-1}i^*$)	0	- 28.5	-27.07	-27.81	-15.60
Amount Paid (F_t)	0	30	30	70	70
Unrecovered Balance (U_t)	-100	- 98.5	-96.57	-54.38	0

Project B2

Time (t)	0	1	2	3	4
Amount Owed (U_{t-1}) ($)	-100	-100	-74.0	-39.2	-22.5
Interest Owed ($U_{t-1}i^*$)	0	- 34	-25.2	-13.3	- 7.5
Amount Paid (F_t)	0	60	60	30	30
Unrecovered Balance (U_t)	-100	- 74	-39.2	-22.5	0

Figure 3-2 can be obtained. Figure 3-2 shows that Project B2, which has a higher rate of return than Project B1, maintains a smaller amount of unrecovered balance than Project B1. In this particular example, it seems that the unrecovered balance appropriately measures the magnitude of actual cash flow patterns. This is because U_t in Equation 3-1 is a direct function of F_t at each point in time.

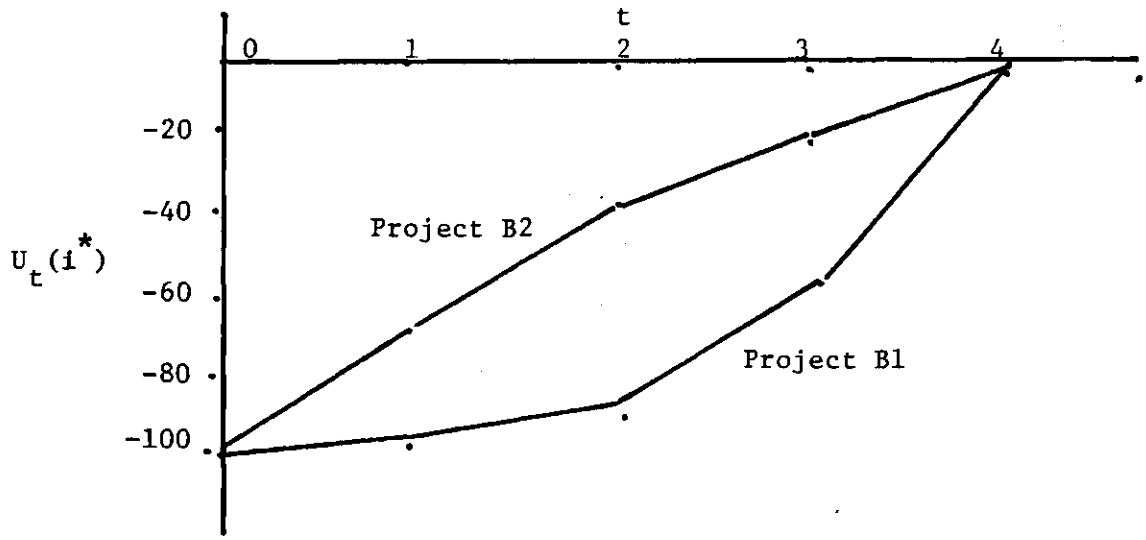


Figure 3-2. The Unrecovered Balance Patterns for Projects B1 and B2

3.1.1.2 Discriminating Ability. The unrecovered balance patterns may take different forms even if the rates of return of the projects are identical. This implies that the identical rate of return for certain projects does not necessarily mean the same pattern of unrecovered balances. This argument is analogous to saying that the identical rate of return for certain projects does not necessarily imply the same economic desirability of the projects. The uniqueness of the unrecovered balance pattern for a given project's cash flows can be ascertained by examining Equation 3-1. In Equation 3-1, it is evident that the rate of return i^* alone does not determine U_t so that one cannot presume that the projects with the same i^* would have the identical values of U_t over time. To illustrate the uniqueness of the unrecovered balance associated with a project, consider Projects C and D having the cash flow patterns shown in Figure 3-3. In order to compute U_t at each point in time for each project, it is necessary to determine the rate of return associated with each project. Both projects have the same rate of return of 28%. However, it becomes evident that the unrecovered balances for each project are quite different, as summarized in Table 3-2 and plotted in Figure 3-4.

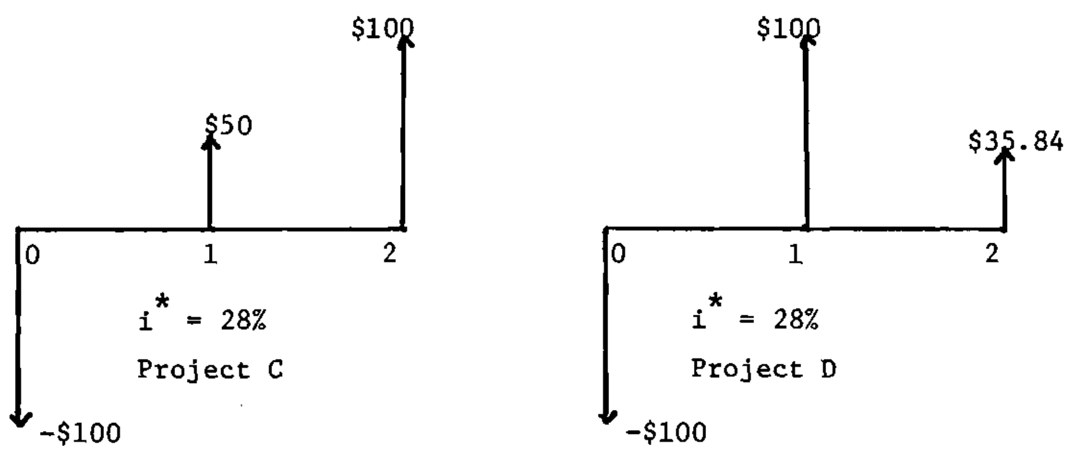


Figure 3-3. Cash Flow Diagrams for Projects C and D

Table 3-2. Unrecovered Balances for Projects C and D

t		0	1	2
$U_t(i^*)$	Project C	-$100	-$78	0
	Project D	-$100	-$28	0

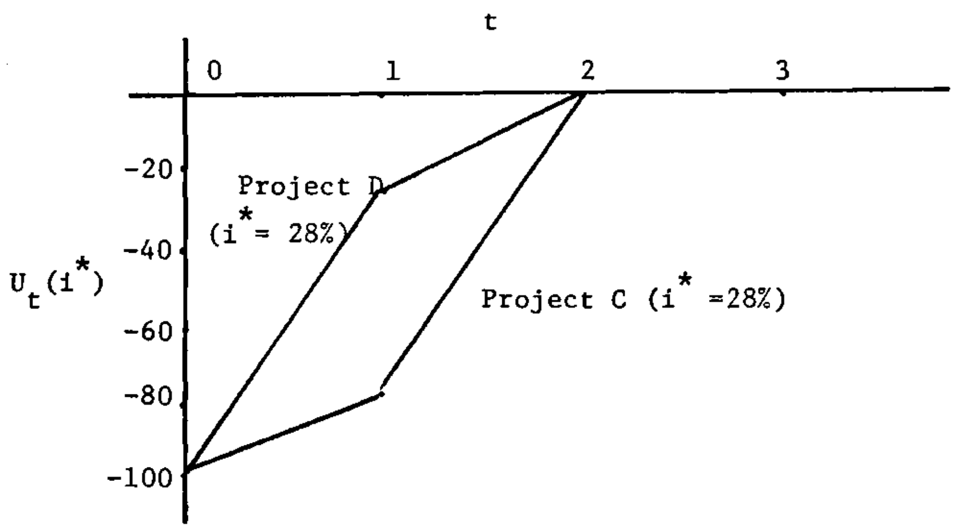


Figure 3-4. Unrecovered Balance Patterns for Projects C and D

3.1.1.3 Limitations of Using Unrecovered Balance as a Measure of Investment Worth. As shown in Equation 3-1, the unrecovered balance pattern appears to be relatively sensitive to the changes in magnitude of cash flows of the project. Thus, it is of interest to examine further whether the concept of unrecovered balance can be utilized effectively as an alternative time-dependent measure of investment worth.

To examine the possibility of using the unrecovered balance concept when measuring the rate at which investment worth is resolved over time, consider the following case of investment situations where Projects E1 and E2 require the same initial outlay of \$100. As can be seen in Figure 3-5, Project E1 only differs from Project E2 in the magnitude of cash flow occurring at the end of year two.

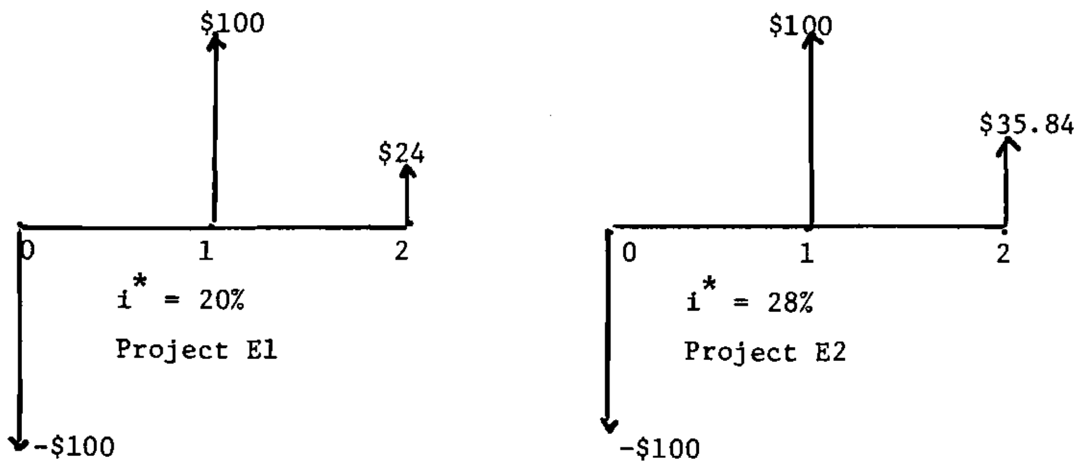


Figure 3-5. Cash Flow Diagrams for Projects E1 and E2

The rate of return associated with each project would be computed as

$$ROR_{E1} = 20\% \qquad ROR_{E2} = 28\%,$$

and the corresponding unrecovered balances as a function of time t can be summarized in Table 3-3.

Table 3-3. Unrecovered Balances for Projects E1 and E2

t		0	1	2
$U_t(i^*)$	Project E1	-$100	-$20	0
	Project E2	-$100	-$28	0

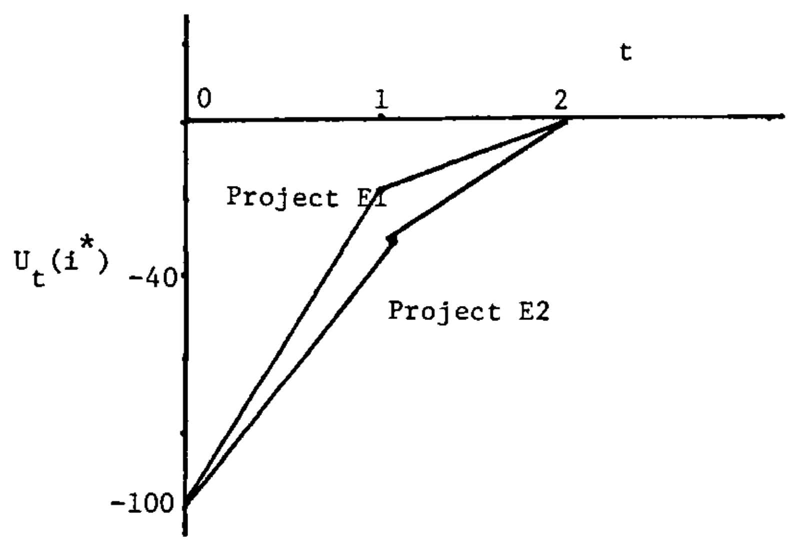


Figure 3-6. The Unrecovered Balance Patterns for Projects E1 and E2

In this example, Project E2 maintains a higher unrecovered balance than Project E1. In terms of the rate at which investment worth is resolved over time, it appears that Project E1 recovers its initial investment at a faster rate than does Project E2. However, by examining the cash flow patterns of both projects (Projects E1 and E2 are common in magnitude of cash flows during time t = 0, 1), it may well be questioned whether Project E1 would be preferred over Project E2 simply because Project E2 maintains a relatively higher unrecovered balance at time t = 1.

The reason for this undesirable property can be explained as follows. In Equation 3-1, i^* (= rate of return) is defined as the interest rate earned on the unrecovered balance at each point in time. Since the unrecovered balance computation is based on the rate of return, i^*, the unrecovered balance at each point in time implicitly assumes the cash flows to be reinvested at i^* (see Heebink [38]). Therefore, it is observed that the implicit assumption has been made that the $100 throw-off of Project E1 in year one is reinvested at 20%, whereas the same amount of $100 throw-off of Project E2 is reinvested at 28%. The use of two different reinvestment rates on the same amount ($100) causes the difficulty experienced in the example.

From the foregoing description it becomes evident that the unrecovered balance does measure the relative magnitude of cash flow for a given investment project. However, if it is used to measure the attractiveness of an investment proposal at the same time, some shortcomings are observed. In other words, it is easy to construct patterns of cash flow which are ranked in one order of merit by the unrecovered balance and in a reverse order by a measure of economic desirability. In view of these conceptual difficulties, the unrecovered balance based on the rate of return for a given project has limitations which reduce its desirability for judging the economic advantage of various investments.

3.1.2 Project Balance as a Time-Dependent Measure of Investment Worth

Another measure of investment worth which reflects the shape of a cash flow pattern is examined in this section. The measure is referred to as the project balance. The fundamental difference between the unrecovered balance and the project balance lies in the fact that the project balance method utilizes a minimum attractive rate of return (MARR) in the computation of unrecovered balance through time, instead of using the rate of return. The use of a single interest rate (reinvestment rate) is to measure investment worth over time on a consistent basis.

3.1.2.1 Project Balance--Basic Definition. The project balance at the end of period t at a MARR of i is defined as the amount the firm has invested in the project or has required from the project at the end of period t, if the outstanding balance at the end of each period 0, 1, 2, ..., t-1 is compounded at interest rate i during the following period. Formally, the balance of a project at the end of period t, at interest i, is

$$S_0(i) = F_0$$

$$S_t(i) = (1+i)\ S_{t-1}(i) + F_t \qquad (3\text{-}2)$$

where $S_t(i)$ = Project balance at end of year t

i = MARR

F_t = Receipts in year of t, and

$S_N(i)$ = Final balance of the project.

It follows immediately that

$$S_t(i) = F_0(1+i)^t + F_1(1+i)^{t-1} + \ldots + F_t$$

$$\text{for } t = 0,1,\ldots,N.$$

Now, it can be seen that the use of the project balance as a basis for determining the desirability of an investment proposal will resolve the inconsistency introduced by the use of the unrecovered balance in the example shown in Figure 3-5. The computations obtained from application of Equation 3-2 for Projects E1 and E2 would produce the following statistics, using a MARR = i.

For Project E1,

$$S_0(i)_{E1} = -100$$

$$S_1(i)_{E1} = S_0(i)_{E1}\ (1+i) + 100 = -100\ i$$

$$S_2(i)_{E1} = S_1(i)_{E1}\ (1+i) + \ 24 = -100\ i\ (1+i) + 24$$

For Project E2,

$$S_0(i)_{E2} = -\ 100$$

$$S_1(i)_{E2} = S_0(i)_{E2}\ (1+i) + 100 = -100\ i$$

$$S_2(i)_{E2} = S_i(i)_{E2}\ (1+i) + 35.84 = -100\ i\ (1+i) + 35.84$$

Since $S_0(i)_{E1} = S_0(i)_{E2}$ and $S_1(i)_{E1} = S_1(i)_{E2}$, it can be said that $S_2(i)_{E2} > S_2(i)_{E1}$ for any value of i ($S_2(i)_{E2} - S_2(i)_{E1} = 11.84$).

Thus, the method of project balance picks E2 over E1, which is consistent with the information on project preference revealed in the example shown in Figure 3-5.

3.1.2.2 Information Provided from Project Balance Pattern. When $S_t(i)$ in Equation 3-2 is plotted as a function of time t over the investment life N, the project balance pattern for a general investment situation can be obtained as shown in Figure 3-7. In Figure 3-7, the following assumptions are made:

1. Cash flows occur at discrete points in time; disbursements are always considered to be made at the beginning of a period and receipts are always received at the end of the period.
2. The discount rate i, or reinvestment rate (MARR), is held constant over the project life. It is recognized that the use of different discount rates over the project life is possible whenever the use of such different interest rates is justifiable. When this is the case, the i in Equation 3-2 can be expressed as a function of time. In this presentation, the use of a single reinvestment rate is assumed unless otherwise specified.

   The discount rate can be viewed as a borrowing or lending rate in a certain investment situation. Although it is recognized that the appropriate selection of a discount rate in constrained capital rationing problems still remains difficult (see Baumol and Quandt [4], Lusztig and Schwab [58], and Thuesen [91]), this study simply utilizes a cut-off rate predetermined by management.
3. The proposal's cash flows are characterized with an initial investment or a series of disbursements, starting at the present followed by a series of positive (or zero) receipts. This assumption will be held throughout the chapter unless otherwise mentioned.

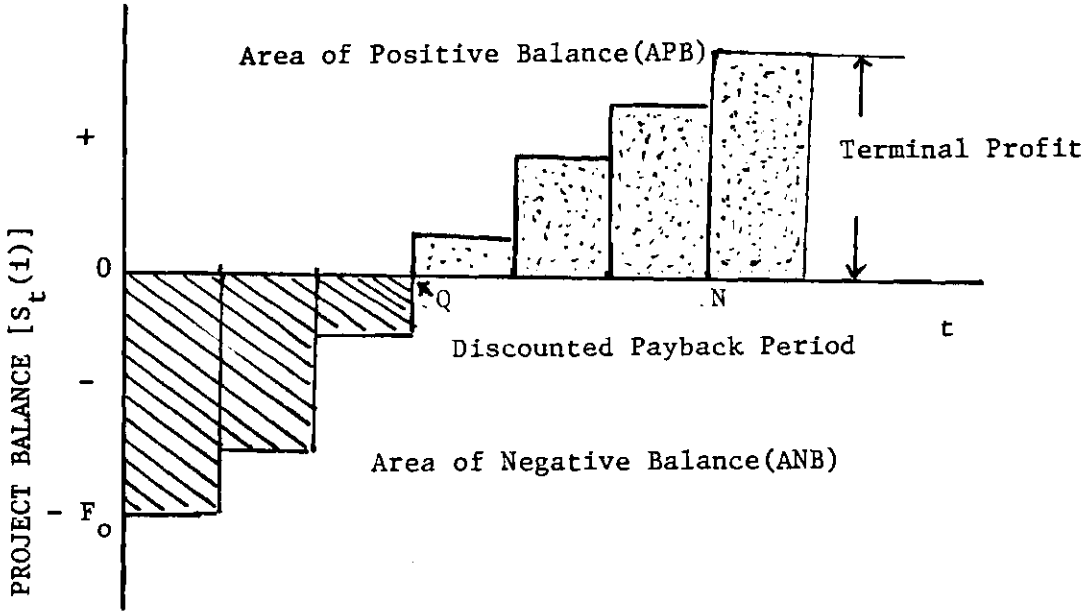


Figure 3-7. Project Balance as a Function of Time

The time path of project balance in Figure 3-7 is referred to as the project balance pattern for the project. This project balance pattern provides the basic information about the attractiveness of a particular investment proposal as a function of its life.

In Figure 3-7, the shaded area represents the period of time during which the $S_t(i)$ has negative values, that is, the time during which the initial investment plus interest is not fully recovered. This area is referred to as area of negative balance (ANB). Symbolically, the area is represented by

$$ANB = \sum_{t=0}^{Q} S_t(i) \tag{3-3}$$

Since the value $S_t(i)$ for $t < Q$ represents the magnitude of negative balance of the project at the end of year t, this is equivalent to the amount of possible loss if the project terminates by that time or no receipts are made. With certainty, the ANB can be interpreted as the aggregate total amount of dollars to be tied up for the particular investment activity. Thus, the smaller the ANB, the more flexible the firm's future investment activity is. Therefore, it is considered that the smaller the ANB for a project, the more attractive it is, if other things are equal.

Point Q on the horizontal axis in Figure 3-7 represents the discounted payback period which indicates how long it will be before the project breaks even. Therefore, it is considered that the smaller the Q for a project, the more desirable it is, if other things are equal.

The dotted area represents the period of time during which the $S_t(i)$ maintains positive project balance. This dotted area is referred to as area of positive balance (APB). The initial investment of the project has been fully recovered so that receipts made during this time period directly contribute to the final profitability of the project. Symbolically, the area is represented by

$$\text{ANB} = \sum_{t=0}^{Q} S_t(i), \quad (Q = \min[q], \text{ and } \sum_{t=0}^{q} F_t(1+i)^{-t}) \tag{3-4}$$

Since the values $S_t(i)$ for $t > Q$ represent the magnitude of positive project balance, there is no possible loss even though the project terminates in a period earlier than its life or no additional receipts are received. Therefore, the larger the APB for a project, the more attractive it is, if other things are equal.

Finally, the last project balance, $S_N(i)$, represents the net future worth of the project (or terminal profit) at the end of its life. The net present worth of the project can be found easily by a simple transformation, that is,

$$PW(i) = S_N(i) \frac{1}{(1+i)^N} \tag{3-5}$$

3.1.2.3 Types of Project Balance Pattern. Numerous types of project balance pattern are feasible depending on the series of cash flows and the interest rate used in the computation of $S_t(i)$. In general, when $S_t(i)$ is plotted as a function of time over the project life, three different types of project balance patterns are possible for the projects which satisfy the basic assumptions outlined in Section 3.1.3.3. The three types of patterns are:

1. When a project provides a return greater than the MARR. This, in turn, indicates that the initial investment of the project is fully recovered with a profit. Then, the general project balance pattern would assume the appearance of Figure 3-7.
2. When a project provides a rate of return equal to the MARR. This implies that the initial investment of the project is barely recovered such that the discounted payback period occurs at the end of the project life. When this is the case, the project balance pattern would assume the appearance of Figure 3-8.
3. When a project provides a rate of return less than the MARR. This implies that the initial investment of the project plus interest is not recovered when the project terminates. The general project balance pattern is described in Figure 3-9.

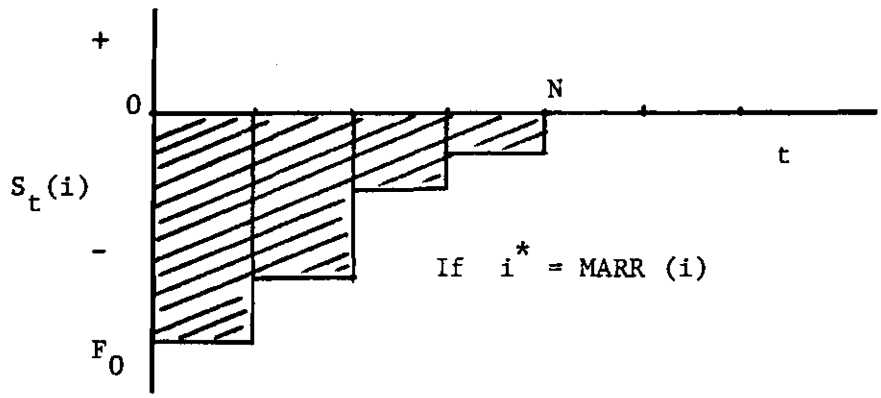


Figure 3-8. Project Balance Pattern when ROR = MARR

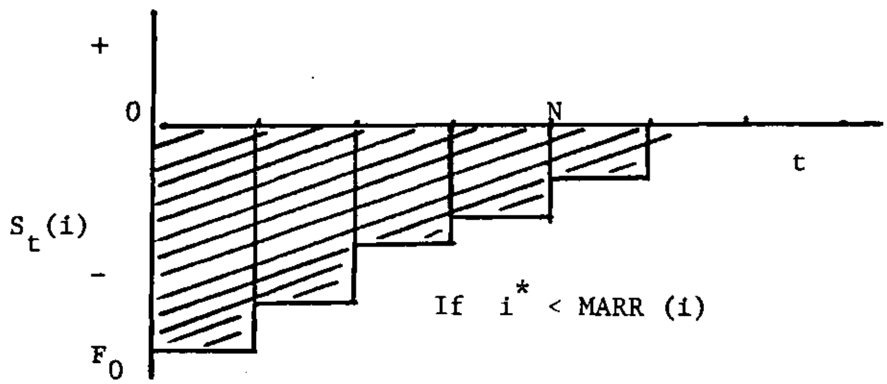


Figure 3-9. Project Balance Pattern when ROR < MARR

3.1.2.4 Discriminating Ability--Regular Periodic Payment Cash Flows. As developed in the previous sections, the project balance pattern provides four different pieces of information regarding the desirability of a proposal as a function of its life. In order to examine the possibility of using the area obtained from the project balance pattern as a basis for measuring uncertainty resolution, the reasoning for this approach (area transformation) is presented in this section. In particular, the types of cash flow patterns examined in this section are regular periodic payment cash flows. The regular periodic payment cash flows are referred to as those cash flow patterns whose amounts are received regularly with equal distance in timing. Although there are an infinite number of feasible cash flow patterns, it is reasonable to expect that a firm's investment proposals could be approximated by one of the following basic cash flow patterns [92, Chap. 4]. These cash flow patterns are:

(1) Single Payment

(2) Uniform Series

(3) Gradient Series (Decreasing)

(4) Gradient Series (Increasing)

One of the basic questions to be answered is as follows: "Is there any loss of information by summarizing the project balance characteristics as ANB and APB?" In other words, can the area transformations ANB and APB be appropriate time-dependent measures of investment worth which provide the same information about the actual cash flow patterns that is reflected in the project balance pattern? To answer this question, consider a project whose cash flow pattern is described as a single payment as shown in Figure 3-10. For a given interest rate i, the project balance pattern at each point in time, $S_t(i)$ can be described as in Figure 3-11.

Since there are no receipts during the project life (i.e., $F_t = 0$, for $t = 1,2,\ldots,N-1$), the $S_t(i)$ of negative project balance at each point in time keeps increasing until the final payment is made. The values of ANB and APB can be found from the recursive equation (see Equation 3-2). That is, by expanding the $S_t(i)$ function over the project life (N), the project balance at each point in time can be expressed as

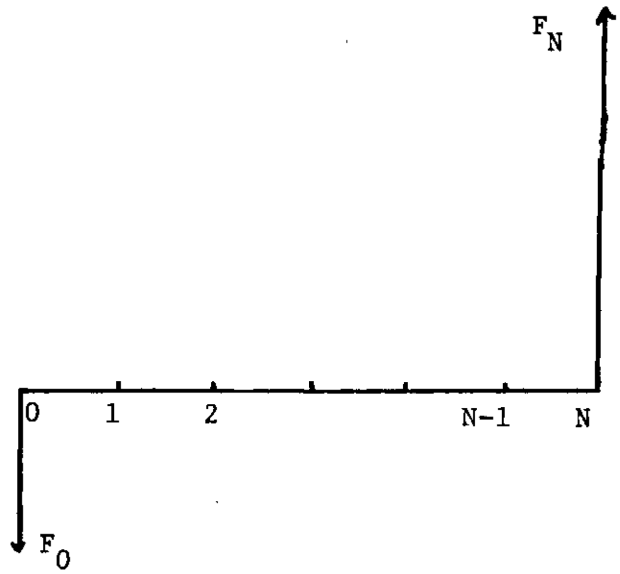


Figure 3-10. Single Payment Cash-Flow Pattern

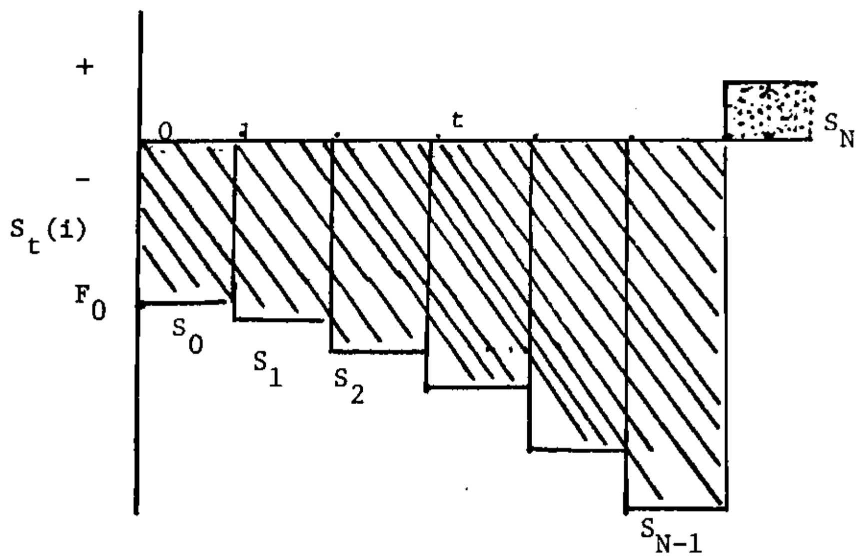


Figure 3-11. $S_t(i)$ over Time for Single Payment

$$S_0(i) = F_0$$

$$S_1(i) = (1+i)F \;\; + F_1$$

$$S_2(i) = (1+i)^2 F_0 + (1+i)F_1 + F_2$$

$$S_3(i) = (1+i)^3 F_0 + (1+i)^2 F_1 + (1+i)F_2 + F_3$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$S_N(i) = (1+i)^N F_0 + (1+i)^{N-1} F_1 + (1+i)^{N-2} F_2 + \ldots + F_N$$

Then, by letting r = (1+i), the values of ANB and APB can be obtained from successive application of the geometric series (see [88]).

$$ANB = \sum_{t=0}^{N-1} S_t(i) = F_0 \left[\frac{1-r^N}{1-r}\right] \tag{3-6}$$

$$APB = \sum_{t=N-1}^{N} S_t(i) = F_0 \left[\frac{1-r^{N+1}}{1-r}\right] + F_N + F_0 \left[\frac{1-r^N}{1-r}\right] \tag{3-7}$$

$$= F_0 r^N + F_N$$

From Equation 3-6 and Equation 3-7, the values of ANB and APB for a single payment cash flow will be uniquely determined as long as the values of i and N are fixed. In other words, ANB and APB become only a function of F_0 and F_N for constant values of i and N. Therefore, for a given cash flow, its corresponding project balance pattern is unique and the area under the project balance pattern becomes unique. This, in turn, implies that there will be no loss of information by summarizing the project balance characteristics as ANB and APB. A similar argument can be applied to those regular cash flow patterns such as uniform series, gradient series (decreasing), and gradient series (increasing). (See Appendix A for the argument.)

To illustrate the basic concept of the project balance and its discriminating ability as compared to the traditional measures of investment worth (present worth, etc.), the following hypothetical investment situation in which the decision maker has to select one of the four proposals shown in Figure 3-12 is presented. Projects 1 and 2 have single payment and uniform series cash flows, respectively. Projects 3 and 4 are gradient series, with one being an increasing gradient series and the other a decreasing gradient series. All projects require the identical initial investment with a service life of three years. If the future worths are computed at a MARR of 10%, all projects would have an equivalent future worth of $63.40 (or present worth = $47.63). This implies that no project is preferable to the others when making a present worth comparison.

Plotting the project balance pattern for each project provides additional useful information which is not revealed by computing present worth equivalents. For example, when comparing Project 1 to Project 3 in terms of the shape of the project balance pattern, it is shown that Project 3 recovers its initial investment within two years, while Project 1 takes three years to recover the same amount of initial investment. This, in turn, indicates that Project 3 would provide more flexibility in future investment activity to the firm as compared to Project 1. By selecting Project 3, the investor can assure himself of being restored to his initial position within a short span of time. A similar one-to-one comparison can be made among all four projects. Table 3-4 summarizes the statistics obtained from each project balance pattern shown in Figure 3-12.

Table 3-4. Statistics of Project Balance Patterns for Projects 1, 2, 3 and 4

Project No	Cash-Flow Pattern	Future Worth FW(10%)	ANB	APB
1	Single Payment	\$63.4	331.00	63.40
2	Uniform Series	\$63.4	150.63	67.07
3	Gradient Series (Decreasing)	\$63.4	141.27	76.73
4	Gradient Series (Increasing)	\$63.4	166.00	63.40

Table 3-4 shows that Project 3 appears to be most desirable even though its terminal profitability is equal to those of other projects, since its ANB is the smallest and its APB area is the largest as compared with those of the other three projects. As discussed in Section 3.1.2.2, the smaller value of ANB implies more flexibility in the firm's future investment activity. In other words, an early resolution of the negative project balance would make funds available for attractive investment opportunities which would arise in the subsequent decision periods. One-to-one comparison among the projects with regard to ANB and APB can be depicted graphically (see Figure 3-13). From Figure 3-13, it becomes evident that the project balance parameters such as ANB and APB reflect the changes in the cash flow patterns over time. Since Point 3 in Figure 3-13 represents the highest APB with the smallest ANB, Project 3 appears to be the most desirable.

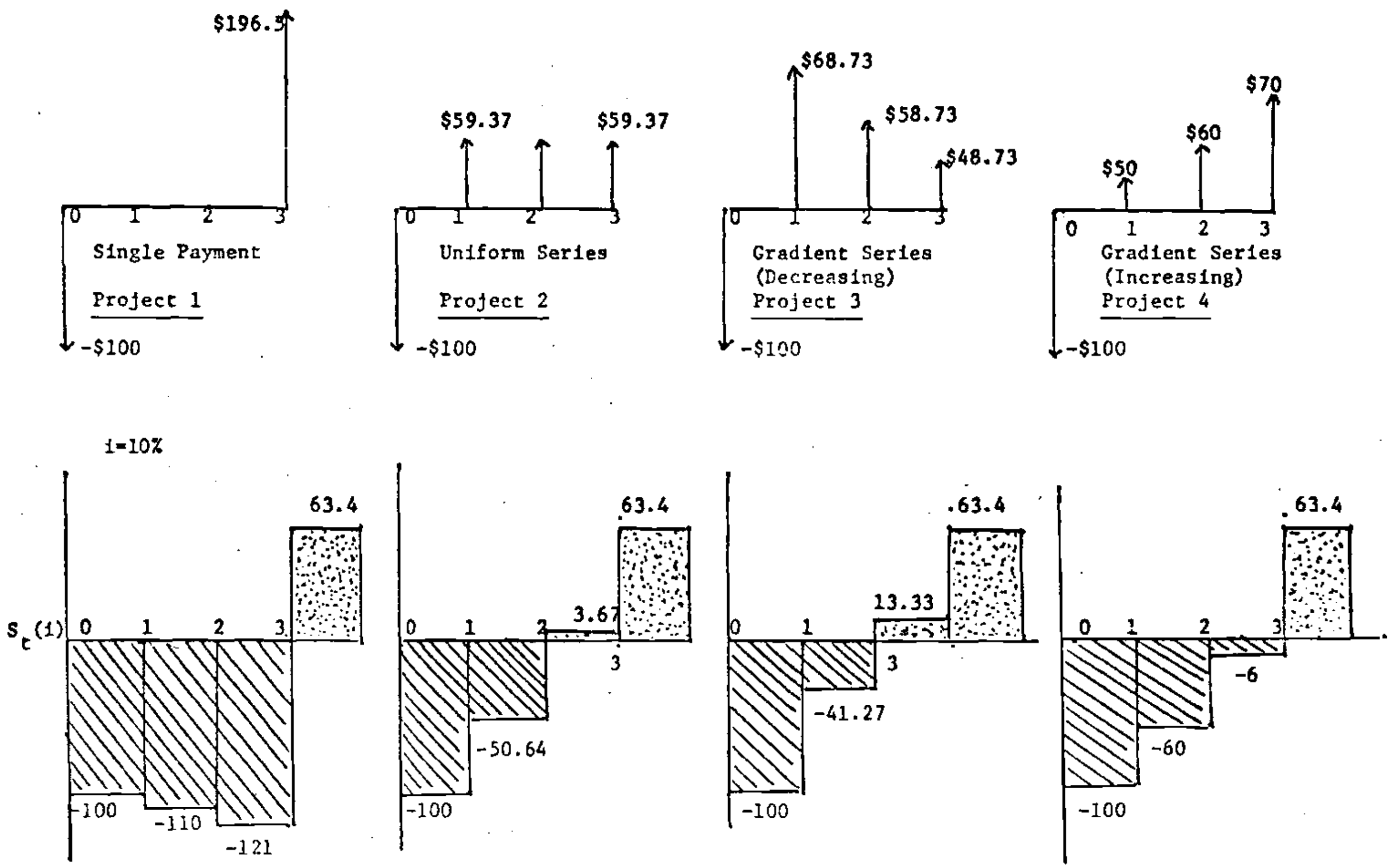


Figure 3-12. Cash-Flow Patterns and Project Balance Patterns for Projects 1, 2, 3 and 4

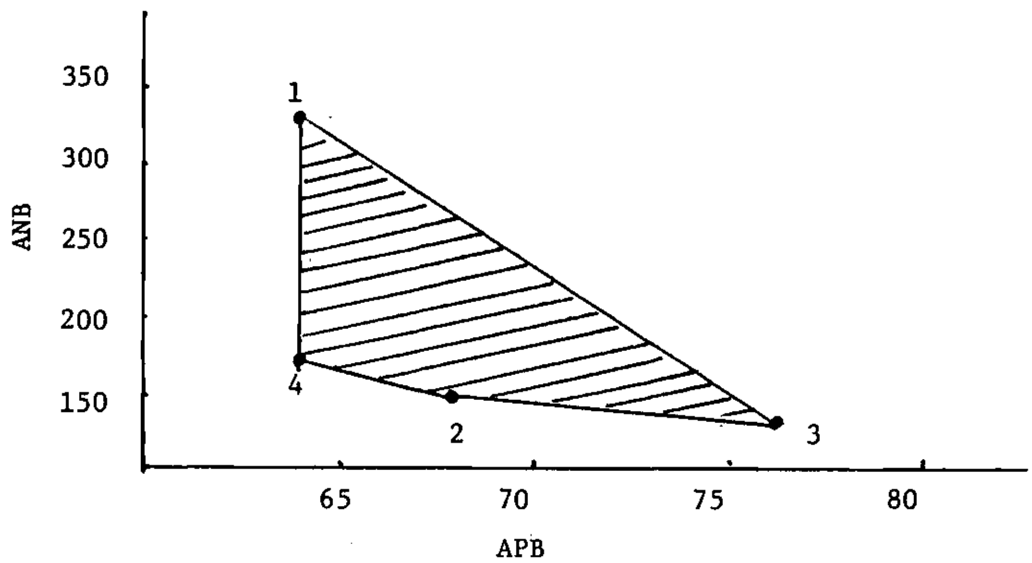


Figure 3-13. Relationship between APB and ANB for Different Cash Flow Patterns

3.1.2.5 Discriminating Ability--Irregular Cash Flow Payments. In the previous section, it was shown that the project balance parameters can provide information regarding the shape of cash flow patterns without loss of any information when the projects have regular, periodic payment cash flows. Although a large portion of all investment proposals can be approximated by one of the regular periodic payment cash flows as mentioned in Section 3.1.2.4, it is also possible to imagine that some proposals do not belong to any class of regular periodic payment cash flows. When this is the case, the proposals' cash flows are said to be irregular periodic payment cash flows.

It is true that even when projects have irregular periodic payment cash flows, the area under the project balance pattern still can be an adequate time-dependent measure of investment worth as compared with the unrecovered balance. There are two reasons for supporting the argument.

First, when the cash flow does not meet the requirement of Assumption 3 in Section 3.1.2.2 (for example, when the cash flow consists of either all receipts or all disbursements with the initial receipt or disbursement occurring at the beginning of year 1), a meaningful rate of return ($-1 < i^* < \infty$) may not exist for the type of cash flow. If this is the case, it is not possible to compute the unrecovered balances for this type of irregular periodic payment cash flow.

The second reason would be that there may be multiple rates of return for the type of cash flow of Project F1 in Figure 3-14. When the proposal has multiple rates of return, it is not clear which rate of return should be used in the computation of the unrecovered balances for the project. However, the project balance method utilizes a single predetermined interest rate to compute the project balances at each point in time so that a unique project balance pattern will always result for any type of irregular periodic payment cash flow.

To examine the effects of the irregular cash flows on the project balance pattern, consider a proposal which does not meet the requirement of Assumption 3 in Section 3.1.2.2 as shown in Figure 3-14 (Project F1). Project F1 violates the basic assumption because the project's cash flow does not have a single initial disbursement or a series of disbursements followed by a series of receipts. In other words, Project F1 has another disbursement occurring at the end of year two, where there is a positive receipt occurring at the end of year one for the same proposal. Is there any difficulty in summarizing the project balance pattern by ANB and APB for this type of proposal?

To answer the question above, consider the following two investment situations shown in Figure 3-14. Project F1 does not resemble any cash flow pattern described in Section 3.1.2.4.

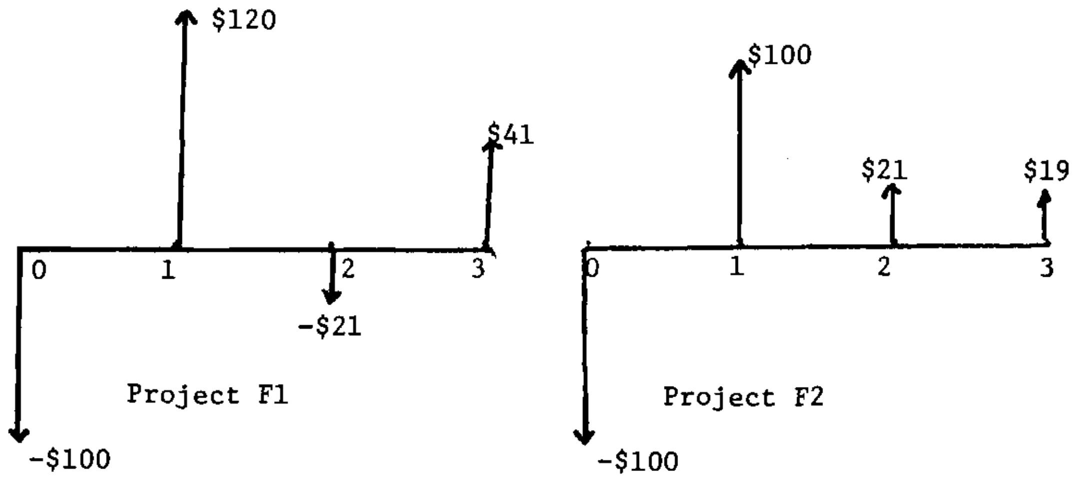


Figure 3-14. Cash Flow Diagrams for Projects F1 and F2

Area transformation in Figure 3-15 indicates no preference between Project F1 and Project F2. However, the project balance pattern for Project F1 indicates that Project F2 recovers more than its initial investment by the end of year one. However, it requires an additional investment in year two such that its profit accumulation rate becomes negative at the end of year two. For Project F2, it may take two years to recover its initial investment, but once recovered, its profit accumulation is increasingly positive for the remainder of its life. However, the preferred project cannot be easily determined by only examining both of the project balance patterns.

When a possible project abandonment opportunity is considered at the end of year one, Project F1 would be preferred because Project F1 will

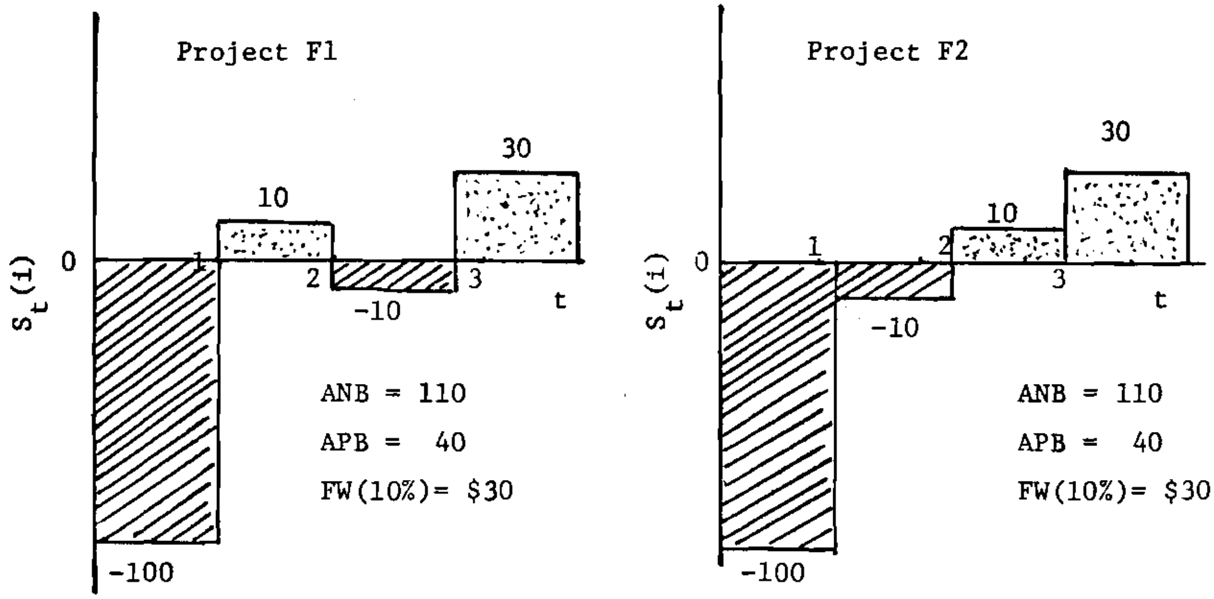


Figure 3-15. Project Balance Patterns for Project F1 and F2

fully recover its initial investment plus interest by that time, while Project F2 will not (see a discussion regarding abandonment decisions in [82]). Thus, the project balance indicates for Project F1 that there would be little loss in final profitability with premature termination. This type of information would be lost by summarizing the project balance characteristics as ANB and APB. However, the project abandonment opportunity is not considered in this study; consequently, such a loss of information in the process of area transformation would not reduce the discriminating ability of the project balance as a time-dependent measure of investment worth.

3.2 Parameters to Be Used in the Measure of Uncertainty Resolution

As pointed out in the beginning of this chapter, the purpose of this research is to explore ways of dealing with the concept of uncertainty resolution in the capital budgeting decision process. In particular, the purpose of this chapter is to examine time-dependent measures of investment worth which provide information necessary to measure uncertainty resolution in Chapter IV.

3.2.1 The Implications of the Two Parameters [$S_N(i)$ and Q]

In Section 3.1.2, a time-dependent measure of investment worth which reflects the shape of the cash flow pattern (the concept of project balance) was introduced. It was also shown that this measure provides information (ANB and APB) in addition to the terminal profitability of an investment. It was also observed that this measure provides information regarding the discounted payoff period (Q). However, with the four information elements available, it becomes evident that to measure uncertainty resolution only two information elements (ANB and APB) reflect the changes in the shape of cash flow pattern with relative sensitivity.

Van Horne uses the terminal profitability as a basis to measure uncertainty resolution [95]. The difficulty of using terminal value as a basis for measuring uncertainty resolution in Van Horne's formula already has been discussed in Chapter II. The parameter $S_N(i)$ is not utilized as a basis to measure uncertainty resolution in this study. However, as will be seen in Chapter V, this information is incorporated into the development of the PB criterion as a normalizing factor.

The use of Q alone (discounted payoff period) as a basis to measure uncertainty resolution also fails to recognize the cash flow pattern even though it generally reflects the magnitude of cash flows in its earlier life of the proposal. Without considering the time value of money (i=0), the measure based on the payoff period is exactly equivalent to Weingartner's approach, which also was discussed in Chapter II. Even for i greater than zero, a similar difficulty associated with Weingartner's measure will be found (see Section 2.3.1).

The difficulty associated with using $S_N(i)$ or Q as a basis to measure of uncertainty resolution can be easily understood from Figure 3-12. As an example, consider Projects 2 and 3 in Figure 3-12. Projects 2 and 3 have the same terminal profitability with the same time period to recover their initial investment at two years. However, Project 2 differs from Project 3 in that Project 3 has a decreasing series of cash flows, while Project 2 has a uniform series of cash flows. This difference in cash flow pattern is reflected in the computation of the ANB for each project ($ANB_{Project\ 2} = 150.64$, $ANB_{Project\ 3} = 141.27$); in turn, a faster recovery of its initial investment is indicated for Project 3 than for Project 2.

3.2.2 Two Parameters (ANB and APB) as a Basis for Measuring Uncertainty Resolution

As seen in Section 3.2.1, only two pieces of information obtained from the project balance pattern can be utilized effectively as a basis for measuring uncertainty resolution. The parameters, ANB and APB, do reflect the shape of cash flow patterns more accurately than do the other two parameters. When ANB and APB are utilized separately as a basis for measuring uncertainty resolution, it is possible to generate two different information elements regarding uncertainty resolution about the same proposal.

A measure of uncertainty resolution based upon ANB statistics would provide information regarding the rate at which the firm's uncertainty about the profit accumulation toward the expected terminal profit is resolved through time. These two measures provide different types of information regarding the desirability of a proposal as a function of time. This information should be evaluated in the light of the firm's future investment opportunities.

CHAPTER IV

MEASURE OF UNCERTAINTY RESOLUTION BASED ON PROJECT BALANCE

In Chapter II, it was pointed out that the use of terminal value as a measure of investment worth fails to consider fully the shape of the cash flow pattern over the project life and therefore cannot be used as a basis for measuring uncertainty resolution. Thus, in Chapter III, a new measure of investment worth which reflects the relative magnitude of the cash flow pattern over time was presented; as a result, the project balance concept was introduced and its uniqueness as compared to the traditional measures of investment worth was discussed.

As a statistical measure of uncertainty resolution in the evaluation of both single risky investments (probabilistic cash flows) and in portfolios of risky investments, the coefficient of variation approach based on terminal value has been utilized [95]. Since the project balance provides more information about the nature of cash flow streams over time, the measure of uncertainty resolution based on the coefficient of variation when the terminal value is replaced by project balance is explored fully in this chapter. Thus, this chapter begins with a discussion of Van Horne's measure of uncertainty resolution and examines how the project balance can be incorporated with the concept of uncertainty resolution.

The limitations of using the coefficient of variation as a measure of uncertainty resolution are discussed and the concept of the expected gain confidence limit [3] is introduced as a means of overcoming this measure's deficiencies. Formally, a measure of uncertainty resolution based on the project balance with the expected gain confidence limit criterion is developed and compared with a measure of uncertainty resolution based on the coefficient of variation using project balance.

4.1 Coefficient of Variation as a Measure of Uncertainty Resolution

Van Horne [95] appears to have been the first to propose an explicit measure for the resolution of uncertainty. His methodology is to use the coefficient of variation as a statistical measure of uncertainty resolution over time (see Section 2.3.2). Therefore, the coefficient of variation at each point in time serves as a relative measure of project variability as time passes and the uncertain future cash flows are realized. The use of the coefficient of variation is tantamount to assuming the standard deviation of the probability distribution of net present values as an appropriate measure of risk.

In Section 4.1.1, an analysis of Van Horne's method of dealing with uncertainty resolution for a single investment is presented. Since the project balance provides more information about the nature of cash flow streams over time, the resolution index based on the coefficient of variation using project balance is explored in Section 4.1.2. The weakness of using the coefficient of variation as a statistical measure of uncertainty resolution is presented in Section 4.2.

4.1.1 Van Horne's Measure of Uncertainty Resolution

In measuring the expected resolution of uncertainty, Van Horne proposes the following statistic as an approximate method of determining relative uncertainty at a moment in time t

$$CV_t = \sigma_t / E\ (NPV) \tag{4-1}$$

where σ_t represents the "weighted" standard deviation of the various branches of the probability tree at the end of period t, and E(NPV) stands for the expected value of net present value at time 0. Thus, the term CV_t represents an "average" coefficient of variation at each point in time.

To illustrate Van Vorne's measure, the example introduced in Section 2-3 is reintroduced. If the firm uses a discount rate of 10% in the appraisal of capital investments, the corresponding probability tree discounted back to time t = 0 using the discount rate would be as shown in Figure 4-1. Since the various payments are converted to their equivalents at the present, it is possible to determine the total present equivalent amount by direct addition. Thus, Figure 4-1 can be transformed into a probability tree with net discounted cash flows occurring only at the branch tips in the terminal period (see Figure 4-2).

As demonstrated by Bierman and Hausman [8], the actual resolution of uncertainty is identical in the two probability trees shown in Figure 4-1 and Figure 4-2. Further, Bierman and Hausman's approach is a more efficient way to compute the same values of CV_t than the one proposed initially by Van Horne. Thus, the former approach is adopted in this presentation.

The expected value of net present value of the project and variance associated with this example would be computed as follows:

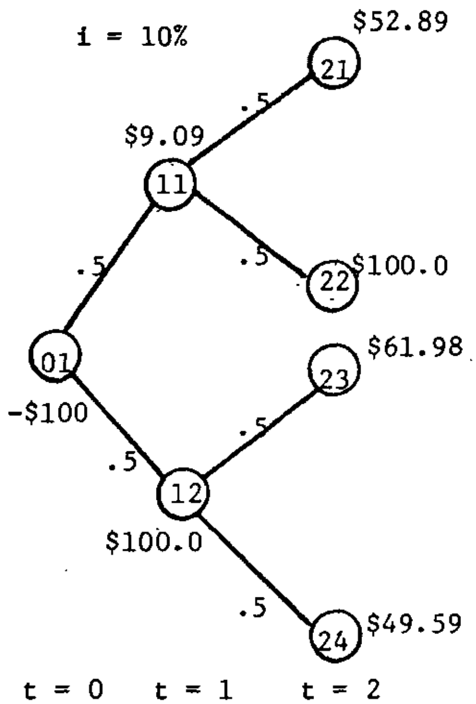


Figure 4-1. Discounted Probability Tree - Van Horne's Example

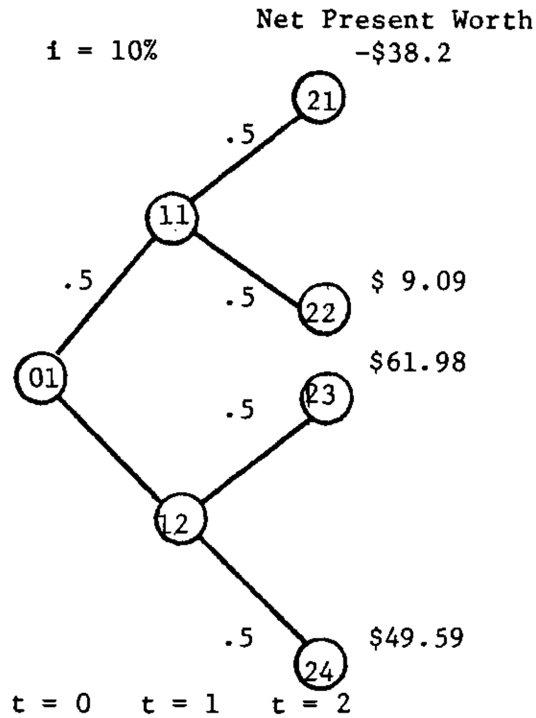


Figure 4-2. Alternate Probability Tree Suugested by Bierman and Hausman

$$E(NPV)_o = (-38.02 + 9.09 + 61.98 + 49.59)\ (.25) = 20.66$$

$$\begin{aligned} VAR(NPV)_o &= (-38.02 - 20.66)^2(.25) + (9.09 - 20.66)^2(.25) \\ &\quad + (61.98 - 20.66)^2(.25) + (49.59 - 20.66)^2(.25) \\ &= 1530.37 \end{aligned}$$

$$CV_o = \sqrt{1530.37}/20.66 = 1.8935$$

For t = 1, first compute the conditional variances associated with Node 11 and Node 12 :

$$\sigma_{11}^2 = (-38.02 - (-14.465))^2(.5) + (9.09 - (-14.465))^2(.5)$$
$$= 554.838$$

$$\sigma_{11}^2 = (61.98 - 55.785)^2(.5) + (49.59 - 55.785)^2(.5)$$
$$= 38.378$$

Then, the weighted conditional variance at time t = 1 will be

$$\text{VAR(NPV)}_1 = \sigma_{11}^2(.5) + \sigma_{12}^2(.5)$$
$$= (.5)(554.838) + (.5)(38.378) = 296.608$$

so that $$CV_1 = \sqrt{296.608}/20.66 = .8336$$

Finally for t = 2 there is no remaining uncertainty. That is, the uncertainty arising from the delayed knowledge of the outcome for a proposed investment project would be completely resolved at the end of the project life. Thus at time t = 2, $\text{VAR(NPV)}_2 = 0$ and therefore $CV_2 = 0$.

Given the CV_t for an investment project, Van Horne argues that the expected resolution of uncertainty for that project can be approximated simply by plotting the CV_t over time. This is done in Figure 4-3 for the numerical example above. Since CV_t represents a decreasing function over time, and as stated previously, the major uncertainty concerning the $100 cash flow is resolved by period t = 1, Van Horne's measure seems to accurately reflect the resolution of uncertainty for this example.

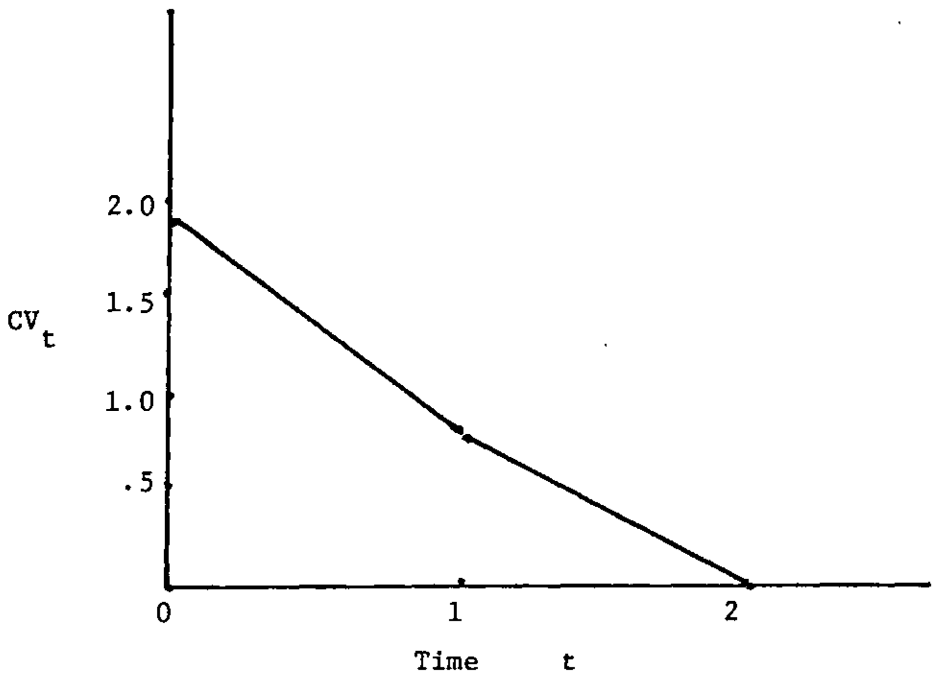


Figure 4-3. Van Horne's Uncertainty Resolution Measure

4.1.2 Coefficient of Variation as a Measure of Uncertainty Resolution When Terminal Value Is Replaced by Project Balance

It was shown in Section 3.1.2 that the project balance provides more information about the nature of a proposal's cash flow stream than the terminal value used as a measure of economic desirability. Since Van Horne's CV_t (V-H's CV_t) is based on terminal value and does not consider the shape of the cash flow pattern, it is of interest to examine how the use of the project balance concept would eliminate the deficiency of Van Horne's approach as observed in Section 2.3.2. When the coefficient of variation is used as an index to measure the rate at which uncertainty about the cash flows is resolved through time (such as proposed by

Van Horne), the following steps would be required to compute the CV_t statistics:

Procedures to Compute CV_t Statistics Based on Project Balance , $\bar{CV}_t$

Step 1: Compute the project balance at each point in time for each branch of the probability tree.

Step 2: Compute the ANB and APB for each branch of the probability tree.

Step 3: Transform the original probability tree into two separate probability trees with the area values (one for ANB and the other for APB) at each branch tip

Step 4: Compute the CV_t statistics for the probability tree with area values with respect to ANB and APB as shown in Section 4.1.

To illustrate the procedure outlined above, consider Projects P1 and P2, which require the same initial investment of $100. Further, they have probability distributions of net present worth with identical expected net present worths and identical variances, using a risk-free discount rate of 10%. A risk-free discount rate is defined as an interest rate which does not consider the effects of risk on present value of a stream of uncertain returns in the time discounting process. The risk-free rate is used as the discount rate in this study because to include a premium for risk in the discount rate would result in double counting with respect to the evaluation of risk (variability).

$$E[NPV]_{P1} = E[NPV]_{P2} = \$15$$

$$\sigma^2[NPV]_{P1} = \sigma^2[NPV]_{P2} = 127.24$$

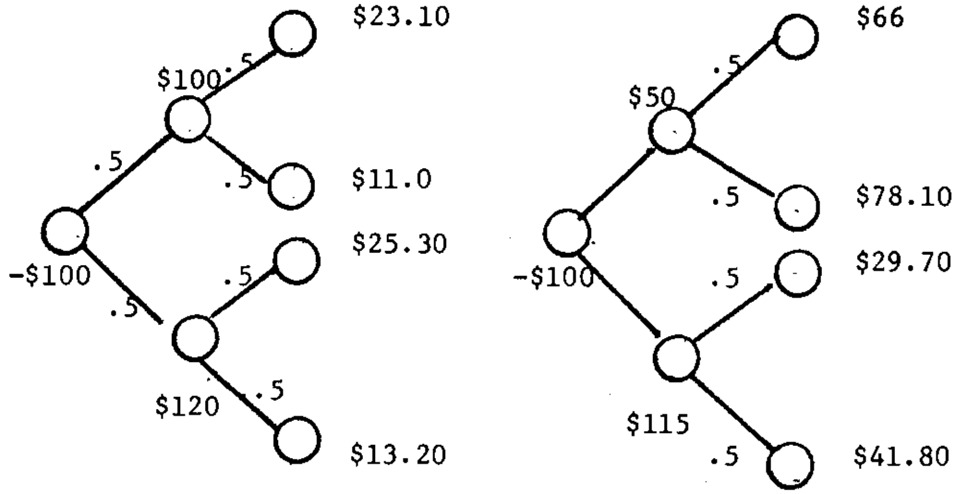


Figure 4-4. Probability Trees Associated with Proposals P1 and P2

Step 1: and Step 2 By utilizing Equation 3-3 through Equation 3-6, the project balances associated with each branch of the probability tree for each alternative are computed as shown in Table 4-1 and their project balance patterns are plotted in Figure 4-4.

Table 4-1. Project Balance at Each Point in Time ($S_t(i)$) For Projects P1 and P2

Branch No.	Project P1 $S_t(i)$ i = 10%				
	t = 0	t = 1	t = 2	ANB	APB
1	-100	-10	12.10	110	12.10
2	-100	-10	0	110	0
3	-100	10	36.30	100	46.30
4	-100	10	24.20	100	34.20
Branch No.	**Project P2 $S_t(i)$ i = 10%**				
	t = 0	t = 1	t = 2	ANB	APB
1	-100	-60	0	160	0
2	-100	-60	12.10	160	12.10
3	-100	5	24.20	100	29.20
4	-100	5	36.30	100	42.30

Step 3: By representing the ANB and APB associated with each alternative as a probability tree with the area values at the branch tips, the probability trees shown in Figure 4-5 are obtained.

Step 4: From application of Van Horne's method, which is outlined in Section 4.1, CV_t statistics for the ANB and APB are computed and summarized in Table 4-2.

Table 4-2. CV_t Statistics for ANB, APB and V-H For Projects P1 and P2

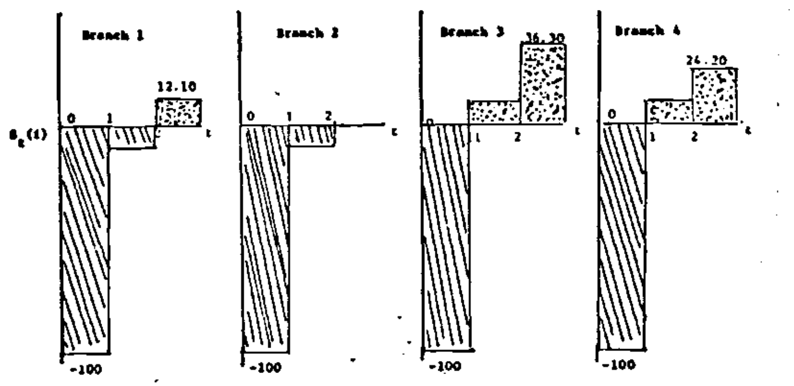
	Project P1			Project P2		
CV_t	Project Balance		V-H*	Project Balance		V-H*
t	ANB	APB	Terminal Value(S_N)	ANB	APB	Terminal Value(S_N)
0	.0476	.7835	.7454	.2308	.7653	.7454
1	0	.3457	.7454	0	.2930	.7454
2	0	0	0	0	0	0

*Van Horne's Resolution Index

As an illustration, the CV_t's for the ANB of Project P1 are computed as follows: From the probability tree associated with APB for Project P1 in Figure 4-6,

$$\text{For } t = 0,\ E[ANB]_0 = 110(.25) + 110(.25) + 100(.25) + 100(.25)$$
$$= 52.5$$

Project P1   i = 10%



Project P2   i = 10%

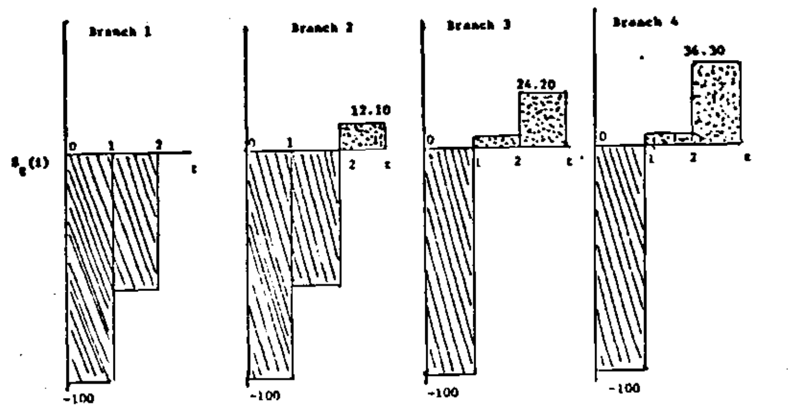


Figure 4-5. Project Balance Patterns Associated with Projects P1 and P2

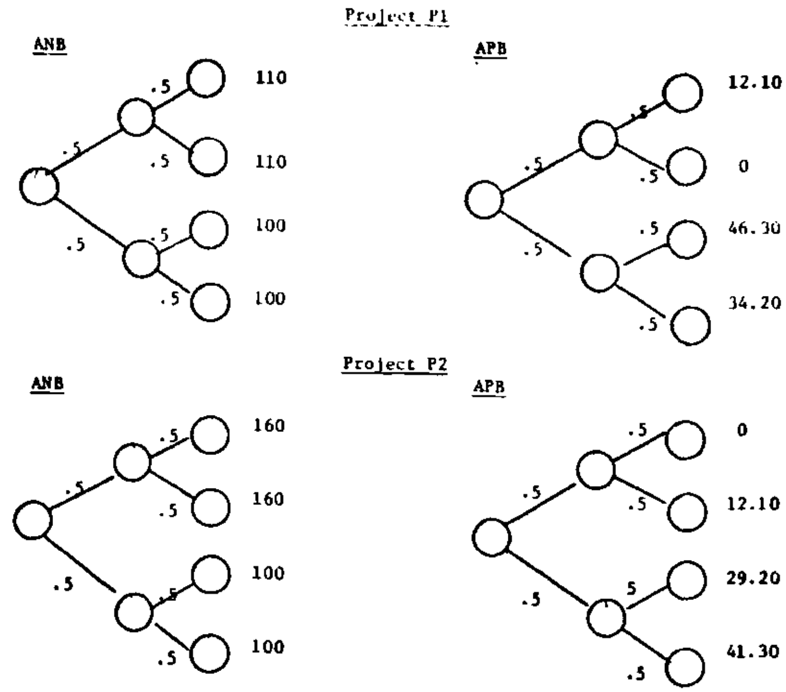


Figure 4-6. Transformed Probability Trees Associated with Projects P1 and P2 Based on ANB and APB

$$\sigma^2[\text{ANB}]_0 = (110\text{-}52.5)^2(.25) + (110\text{-}52.5)^2(.25)$$

$$+ (100\text{-}52.5)^2(.25) + (100\text{-}52.5)^2(.25)$$

$$= 6.245$$

$$C\bar{V}_0 = (\sigma^2[\text{ANB}])^{\frac{1}{2}}/E[\text{ANB}]_0 = .0476$$

For t = 1, first compute the conditional variances with Node 11 and Node 12:

$$\sigma^2[\text{ANB}]_{11} = (110\text{-}110)^2(.25) + (110\text{-}110)^2(.25) = 0$$

$$(\text{since } E[\text{ANB}]_{11} = 110)$$

$$\sigma^2[\text{ANB}]_{12} = (100\text{-}100)^2(.5) + (100\text{-}100)^2(.5) = 0$$

$$(\text{since } E_{12}[\text{ANB}] = 100)$$

Then, the weighted conditional variance at time t = 1 will be

$$\sigma_1^2[\text{ANB}] = \sigma_{11}^2(.5) + \sigma_{12}^2(.5) = 0$$

so that $$C\bar{V}_1 = (\sigma_1^2[\text{ANB}])^{\frac{1}{2}}/E[\text{ANB}]_0 = 0$$

Finally, there is no remaining uncertainty; thus $C\bar{V}_2 = 0$.

The values of $C\bar{V}_t$ over time are presented in Figures 4-7 and 4-8. In Figure 4-7, the expected resolution of uncertainty based upon the terminal profitability (V-H index) indicates that there has been no resolution of uncertainty between period t = 1 and period t = 2. Since $E[NPV]_{P1} = E[NPV]_{P2}$, $\sigma^2[NPV]_{P1} = \sigma^2[NPV]_{P2}$ and V-H $CV_{t,P1}$ = V-H $CV_{t,P2}$ for all t, Van Horne's measure would indicate both projects have an equal effect.

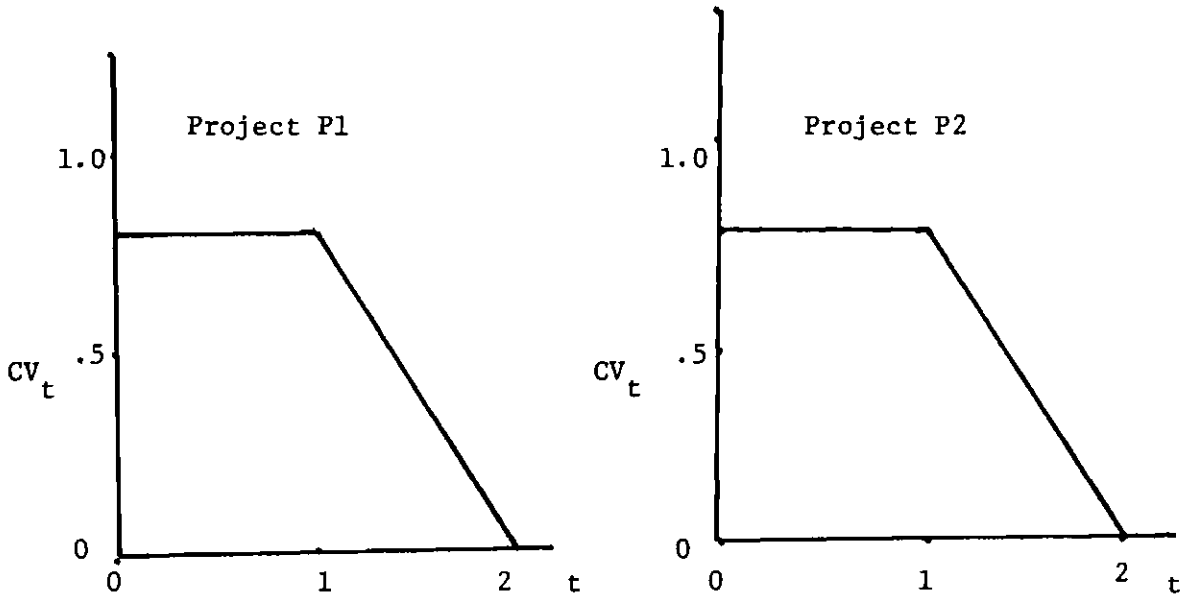


Figure 4-7. Van Horne's Resolution Index for Project P1 and Project P2

It is clear from Figure 4-8 that computations obtained from $\bar{CV}_t$ based on project balance indicate that Project P1 would be preferred if the recovery of the initial investment of a project is a primary concern. This is because the smaller the coefficient of variation for a project the less risk it has according to the definition (for t = 0, $\bar{CV}_1[ANB]_{P1}$ = .0476, $\bar{CV}_1[ANB]_{P2}$ = .2308). Otherwise Project P2 would be desirable if the rate of profit accumulation is of utmost interest (for all t, $\bar{CV}_t[APB]_{P1} > \bar{CV}_t[APB]_{P2}$). In addition, Figure 4-8 also indicates that uncertainty about negative project balance of either project would be completely resolved at the end of year one. Once the initial investment is recovered and profits begin to accumulate (see Figure 4-5), then Project P2 would have less variation about the anticipated realization of future cash flows occurring at the end of year two. Thus, in this

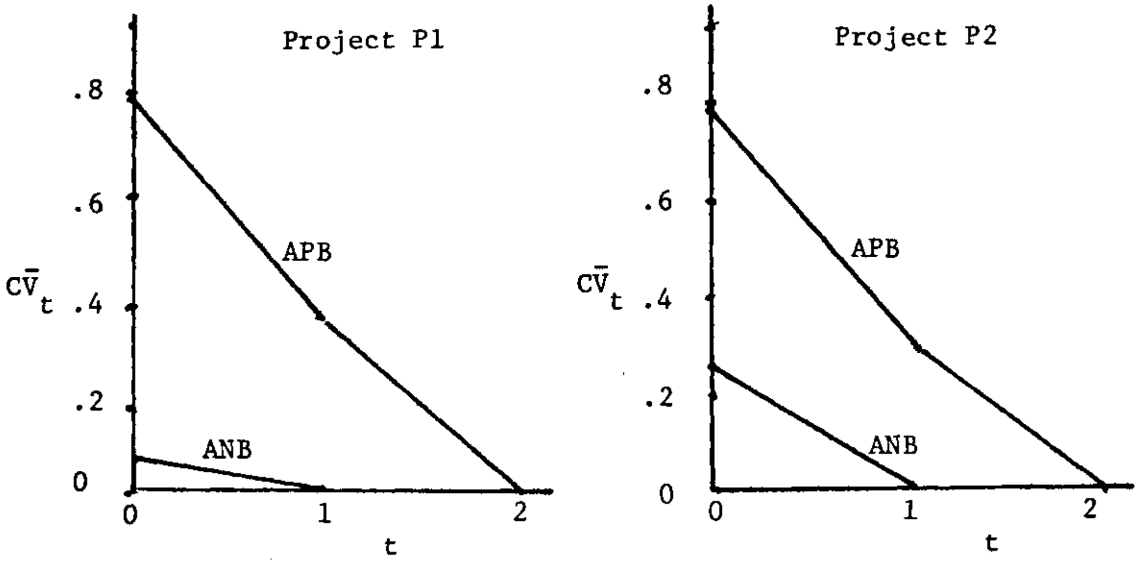


Figure 4-8. Resolution Indexes Based on Project Balance for Project P1 and Project P2

example, the coefficient of variation using project balance rather than terminal value seems to reflect the way that the uncertainty about the project is reduced as its future outcomes are realized. The reason for this is that project balance provides more information about the nature of a proposal's cash flow stream (i.e., it is sensitive to changes in cash flow pattern).

In the previous example, it was shown that the competing projects have the same E(NPV) and σ^2(NPV) with the identical pattern of uncertainty resolution for Van Horne's methodology. However, with project balance substituted in Van Horne's method, it is seen that the way uncertainty is resolved is different for the two projects.

Counter-examples are also possible where under Van Horne's methodology the uncertainty resolution properties V-H's CV_t favor one project over another while the project balance approach would provide identical uncertainty resolution characteristics (both ANB and APB). However, these latter situations are less likely to occur than the previous situations, as evidenced in Appendix B.

As shown in the example (Figure 4-4) and in Appendix B, the weakness of Van Horne's measure becomes evident. Therefore, when the coefficient of variation is selected as a measure of uncertainty resolution, the CV_t statistics based on project balance serve as a more discriminating measure of uncertainty resolution as compared with the CV_t statistics based on terminal value (i.e., Van Horne's measure).

4.2 The Difficulty of Using the Coefficient of Variation as a Statistical Measure of Uncertainty Resolution

In the previous section, some difficulties with Van Horne's CV_t were illustrated and for those cases, use of ANB and APB worked nicely. However, it would be possible to have some examples where Van Horne's method works better than ANB and APB, but those cases were "unlikely" to occur. Thus, the idea is that the basis to be used in the computation of $\bar{CV}_t$ statistics would be project balance rather than terminal value.

As shown in Equation 4-1 the use of coefficient of variation assumes standard deviation as a measure of absolute variability. Standard deviation, however, has several weaknesses as a measure of variability, as discussed by Levy [53]. Therefore, the coefficient of variation inherently reveals the weaknesses similar to those of the standard deviation.

In Section 4.2.1, the problems associated with using the coefficient of variation as a time-dependent measure of uncertainty resolution are presented. A discussion of the deficiency of standard deviation as a measure of variability follows in Section 4.2.2. The advantage of utilizing the expected gain confidence limit proposed by Baumol [3] over the standard deviation as a measure of variability also is discussed.

4.2.1 Problems Associated with Using the Coefficient of Variation in the Measure of Uncertainty Resolution

As shown by Equation 4-1, the coefficient of variation is a measure of relative dispersion derived by dividing the standard deviation of a set of numbers by the arithmetic mean of the set. In other words, this measure considers all variability as adverse variability. In fact, this implication is not acceptable to most risk-averters because as a measure of dispersion it implies to them that deviations below expected value are of no greater concern than deviations above expected value. Thus, the most significant weakness of the CV_t is the fact that it uses the standard deviation in its numerator and it therefore does not distinguish between positive and negative variations.

To illustrate the deficiency of using the coefficient of variation as a time-dependent measure of uncertainty resolution, consider the probability trees shown in Figure 4-9. In Figure 4-9 Projects A_1 and A_2 require an identical initial investment of $100 with a three-year service life. By computing the expected net present worth and the variance about the mean for each project, it is observed that each project has the same expected net present value of $16 and the same dispersion about the mean, VAR(NPV) = 504, using a risk-free interest rate of 10%. The project balance determined by Equation 3-3 at each point in time for each branch is summarized in Tables 4-3 and 4-4, and plotted in Figure 4-10. By transforming each project's original probability tree into one with ANB and APB occurring only at the branch tips in the final period, the probability

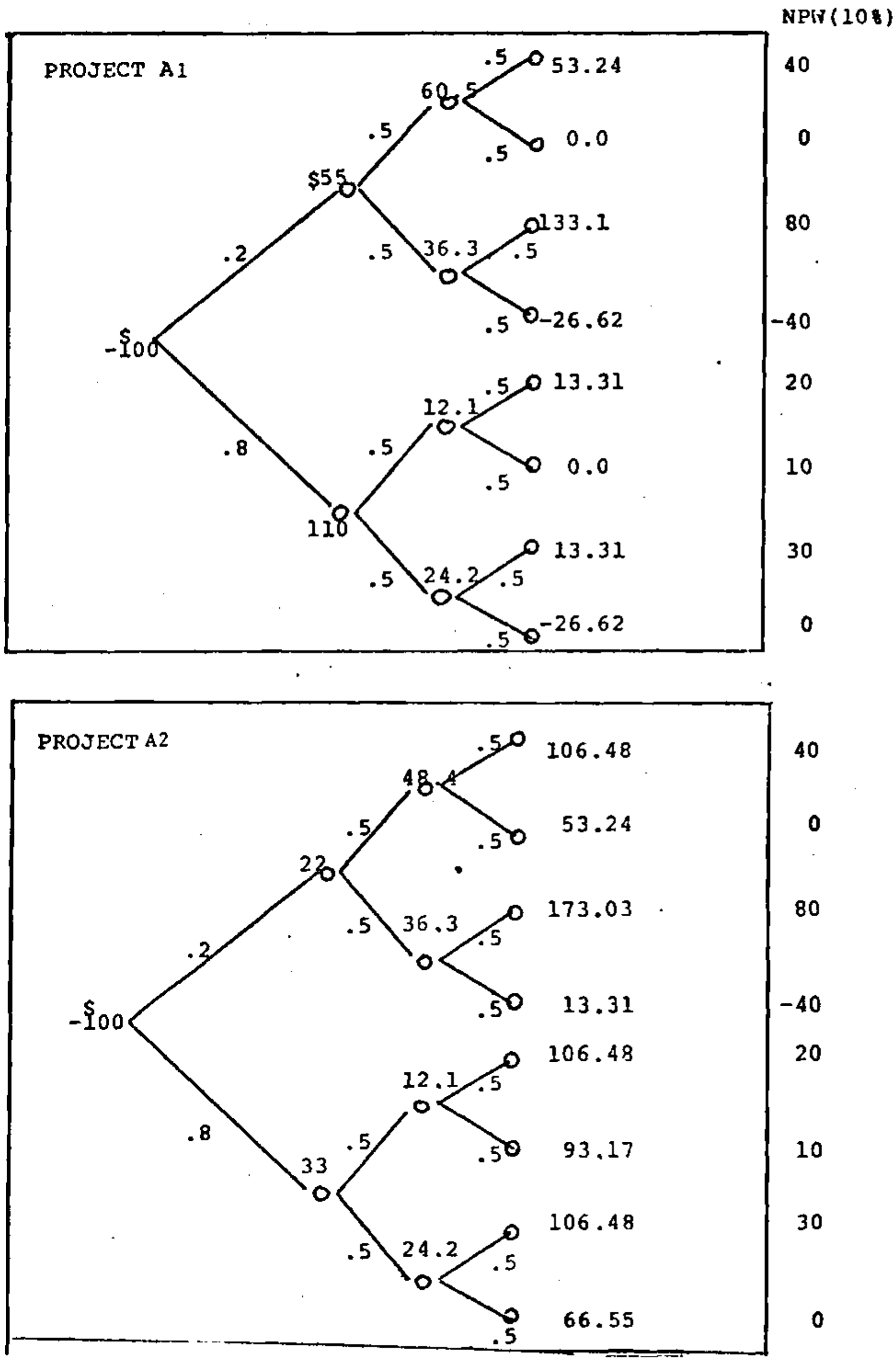


Figure 4-9. Probability Trees Associated with Project A1 and Project A2

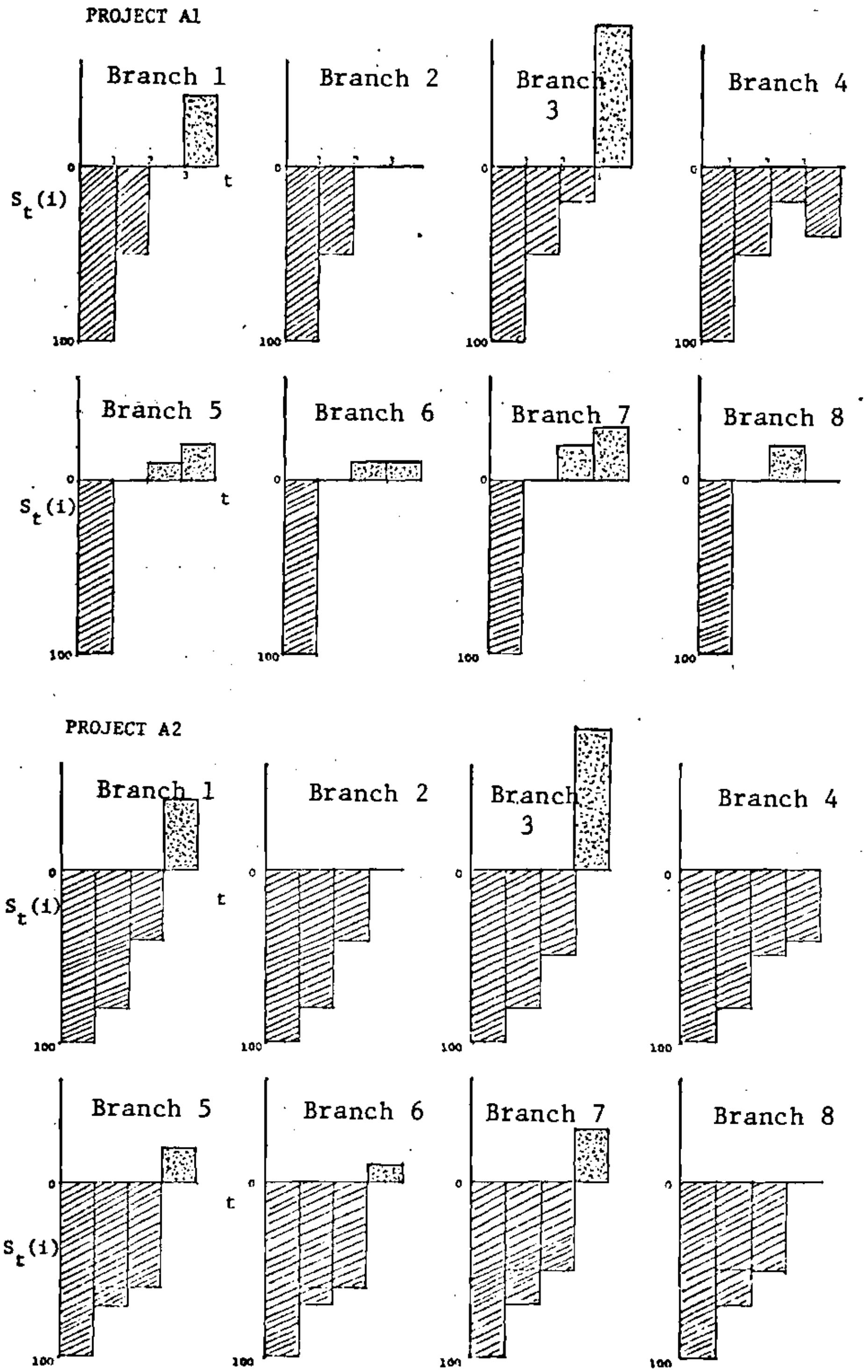


Figure 4-10. Project Balance Patterns Associated with Each Branch of the Probability Trees for Projects A1 and A2

Table 4-3. Project Balance at Each Point in Time ($) (Project A1)

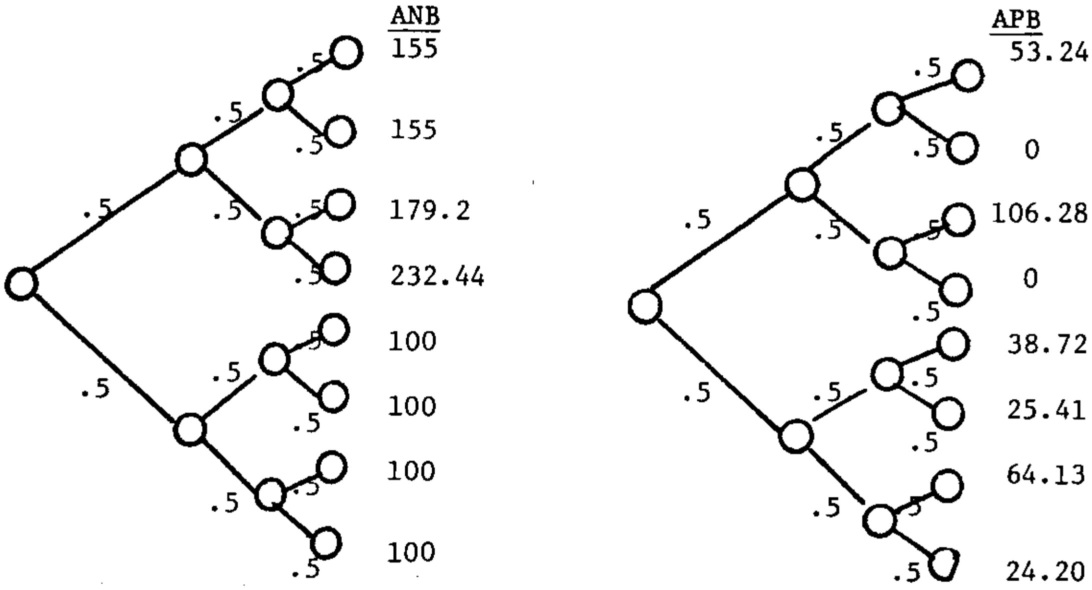
Branch No.	t = 0	t = 1	t = 2	t = 3	ANB	APB
1	-100	-55	0	53.24	155	53.24
2	-100	-55	0	0	155	0
3	-100	-55	24.20	106.48	179.20	106.18
4	-100	-55	-24.20	-53.24	232.44	0
5	-100	0	12.10	26.62	100	38.72
6	-100	0	12.10	13.31	100	25.41
7	-100	0	24.20	39.93	100	64.13
8	-100	0	24.20	0	100	24.20

Table 4-4. Project Balance at Each Point in Time ($) (Project A2)

Branch No	t = 0	t = 1	t = 2	t = 3	ANB	APB
1	-100	-88	-48.40	53.24	236.40	53.24
2	-100	-88	-48.40	0	236.40	0
3	-100	-88	-60.50	106.48	248.5	106.48
4	-100	-88	-60.50	-53.24	301.74	0
5	-100	-77	-72.60	26.62	249.6	26.62
6	-100	-77	-72.60	13.31	249.6	13.31
7	-100	-77	-60.50	39.93	237.5	39.93
8	-100	-77	-60.50	0	237.5	0

trees shown in Figure 4-11 are obtained.

Project A1



Project A2

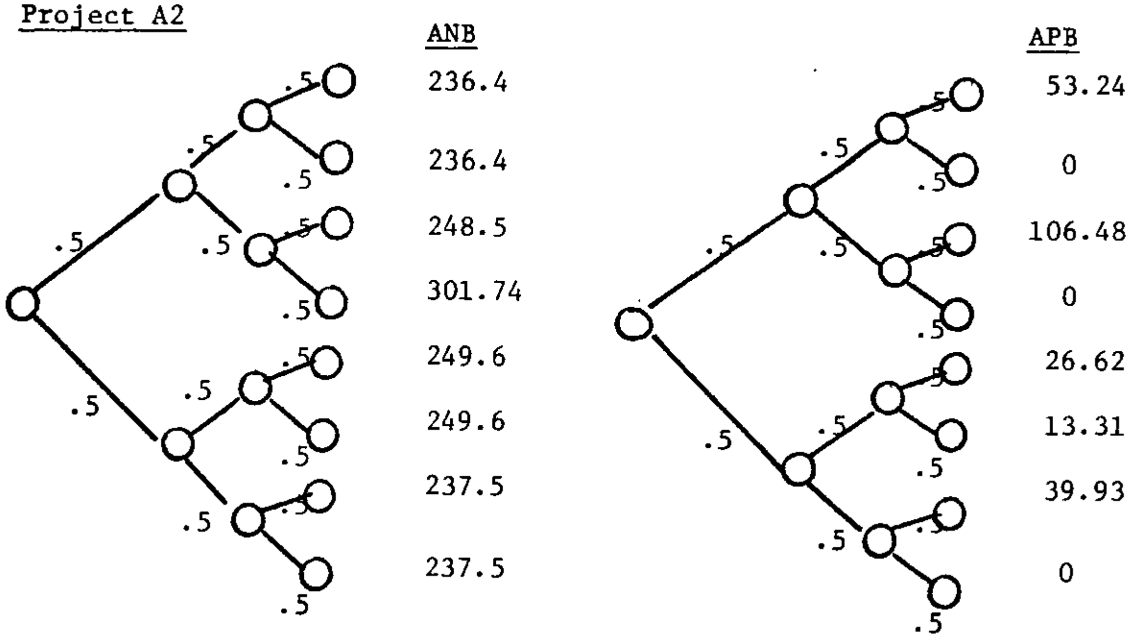


Figure 4-11. Transformed Probability Trees for ANB and APB (Projects A1 and A2)

Once a probability tree is transformed, application of Van Horne's method (V-H's CV_t), which is outlined in Section 4.1.1, would yield the following average coefficients of variation over time for ANB and APB probability trees, respectively. As shown in Table 4-5, Van Horne's measure indicates that both projects have the same pattern of uncertainty resolution. Further, the expected resolution of uncertainty based upon the terminal value fails to note any change in uncertainty from t = 1 to t = 2. This situation will occur whenever the nodes have identical conditional expected returns. Since $E(NPV)_{A1} = E(NPV)_{A2}$ and $VAR(NPV)_{A1} = VAR(NPV)_{A2}$, it may be said that projects A1 and A2 would be equally desirable when evaluated by Van Horne's criterion.

Table 4-5. Resolution Index Based on Coefficient of Variation

	Project A1			Project A2		
	Project Balance		V-H CV_t	Project Balance		V-H CV_t
TIME	ANB	APB	TERMINAL PROFIT	ANB	APB	TERMINAL PROFIT
0	.3027	.6348	1.4031	.0573	1.0482	1.4031
1	.1218	.6346	1.3975	.0538	.9938	1.3975
2	.0725	.5991	1.3975	.0342	.9623	1.3975
3	0	0	0	0	0	0

E[ANB] = 116.082	E[ANB] = 245.992
E[APB] = 38.478	E[APB] = 23.958
E(NET PRESENT VALUE = $16.0	E(NET PRESENT VALUE) = $16.0

On the other hand, computations obtained from application of the project balance in the $C\bar{V}_t$ indicate that Project A2 would be preferred over Project A1 when negative project balance solely is considered (See Figure 4-12), while project A1 would be preferred over Project A2 when positive project balance only is of concern (See Figure 4-13). This is because the smaller the coefficient of variation a project has, the less risk it has, according to the definition. However, a careful examination of Figure 4-10 reveals that most of the uncertainty concerning Project A1's negative project balance would be resolved at the end of year two, while uncertainty about Project A2's negative project balance would not be resolved until the end of year three. In fact, in terms of negative project balance, Project A1 would be the safer one to undertake as compared with Project A2. However, the resolution index as measured by the coefficient of variation as shown in Figure 4-12 does not properly reveal the information contained in the probability tree shown in Figure 9-9.

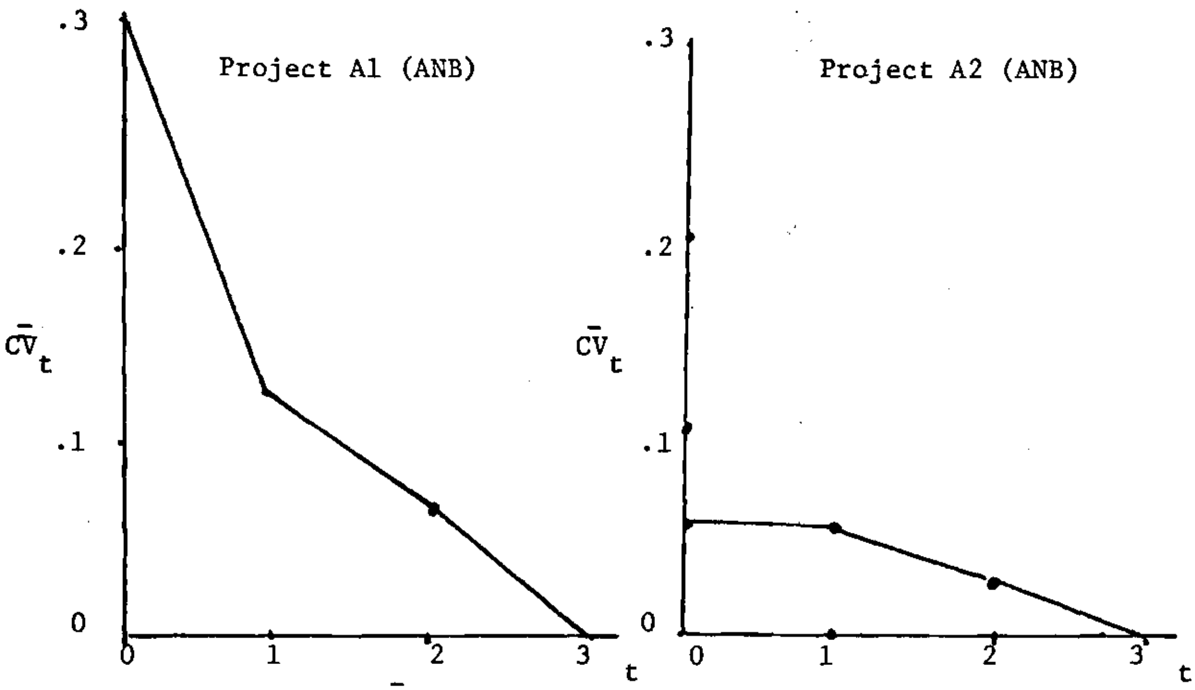


Figure 4-12. $\bar{CV}_t$ of ANB for Projects A1 and A2

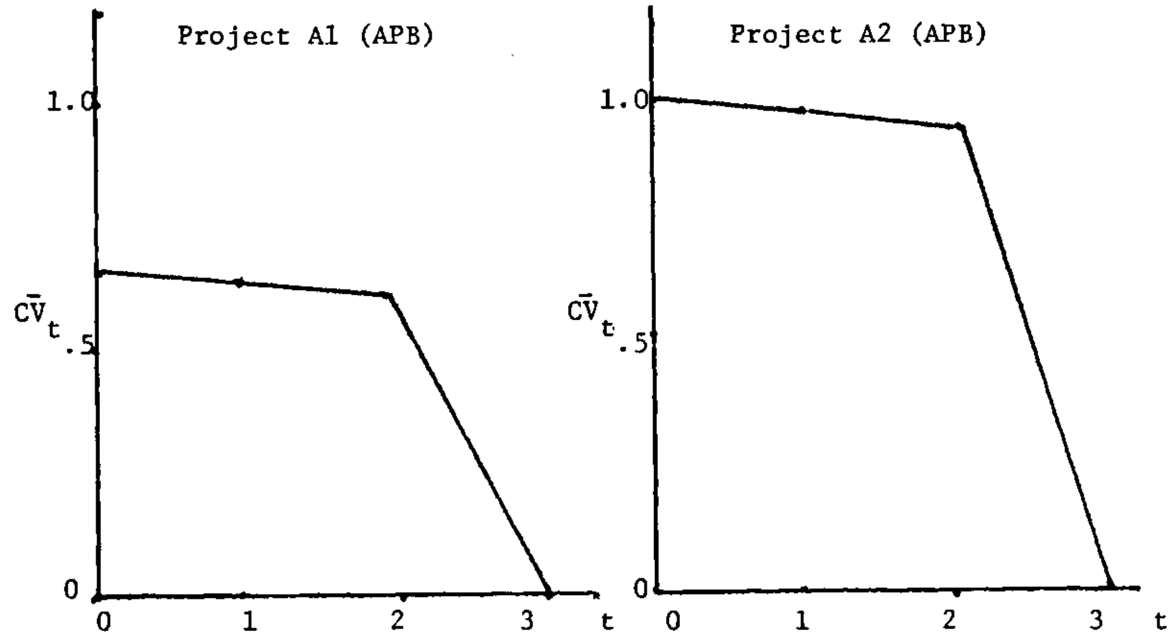


Figure 4-13. Uncertainty Resolution Patterns of APB for Projects A1 and A2

It must be understood that the deficiency shown in the previous example is <u>not</u> attributable to the use of project balance in lieu of terminal value in the computation of CV_t statistics. In other words, the same deficiency will be observed whenever the coefficient of variation is utilized as a measure of uncertainty resolution based on either terminal value or project balance. The reason for this result is discussed in the following section.

4.2.2 The Limitations of CV_t as a Measure of Variability

The limitation of using CV_t as a measure of variability (risk) can be illustrated by considering the following two investment projects, which have the expected net present values and standard deviations shown in

Table 4-6. If the decision maker evaluates investment proposals on the basis of information about the expected value and dispersion of the probability distributions of possible future cash flows, then the coefficient of variation would serve as a relative measure of the degree of uncertainty. In Table 4-6 the coefficient of variation for Project X2, 0.26, is greater than that for Project X1, 0.25. According to this definition, it would be said that Project X2 had the greater degree of uncertainty.

Table 4-6. Probabilistic Data of Project X1 and Project X2

	PROJECT XI	PROJECT X2
$E[NP\bar{V}]$ ($)	800	1,500
σ	200	400
$CV(= \frac{\sigma}{E[\overline{NPV}]})$	.25	.26
$E + 3\sigma$	1,400	2,700
$E + 2\sigma$	1,200	2,300
$E + \sigma$	1,000	1,900
E	800	1,500
$E - \sigma$	600	1,100
$E - 2\sigma$	400	700
$E - 3\sigma$	200	300

To facilitate the discussion, assume that the basic random variable (the return on the investment) for the given example is normally distributed. Then, the probability distribution of net present value for Projects X1 and X2 would be depicted in Figure 4-14. Then it can be said that there is only about a 16% probability that the realized return will ever fall below $E - \sigma$, only a 2% probability that it will ever be lower

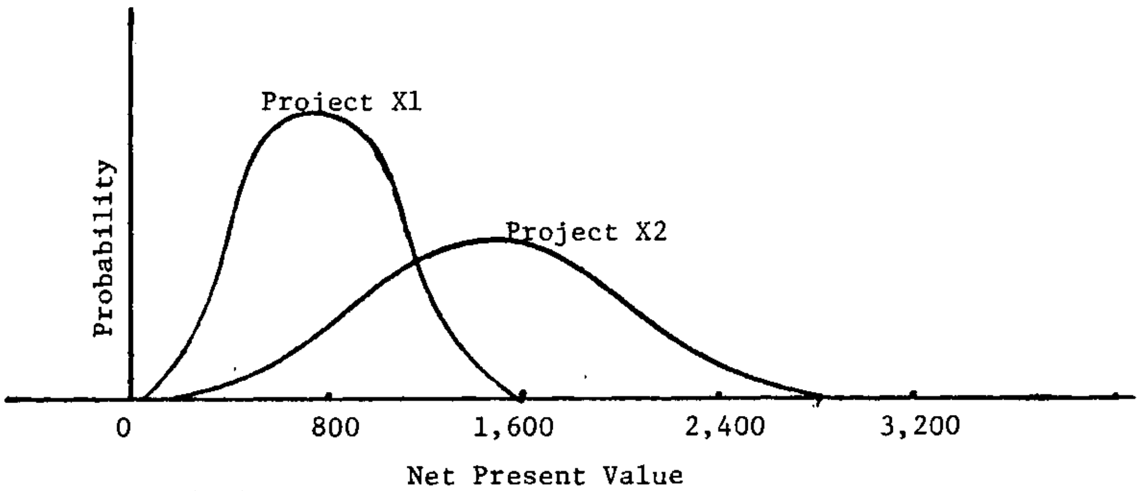


Figure 4-14. Probability Distributions of Net Present Values for Project X1 and Project X2

than E - 2σ, no more than 0.1% probability that it will fall below E - 3σ, and so on. In the example above, for the worst anticipated outcome associated with X2, E - 3σ = 300 is still better than the corresponding outcome associated with X1 (E - 3σ = 200). Now it may well be questioned whether anyone would still choose Project X1 simply because it has a relatively lower degree of variability as compared with Project X2. The difficulty is that σ is not, per se, an adequate measure of risk, for it treats with indifference potential outcomes falling above E and those falling below E. However, by definition, the coefficient of variation uses the standard deviation in its numerator and it therefore does not distinguish between positive and negative variations. In other words, the coefficient of variation would not be an appropriate measure of the uncertainty faced by the investor.

4.2.3 The Expected Gain Confidence Limit as a Measure of Variability

As mentioned in Chapter II, the use of variance as a measure of risk treats with indifference potential outcomes above and below the expected value. Thus, a modified version of the measure is proposed by Baumol, which is referred to as the expected gain confidence limit criterion (EGCL) [3]. This supplementary measure takes the form of $L = E - \delta\sigma$ where E and σ stand for the expected value and standard deviation of the net present value about the mean of an investment proposal, and δ is a parameter which represents the degree of risk aversion of the decision maker. Here, L is said to be the critical point on which an investment decision should be based. As discussed in Section 2.2.1, this point of view on risk is somewhat the way businessmen define risk. Therefore, in this study the concept of the expected gain confidence limit criterion is adopted as a means of overcoming the deficiencies associated with using the coefficient of variation as a measure of uncertainty resolution. Consequently, in Section 4.3, a measure of uncertainty resolution based on the expected gain confidence limit criterion will be developed.

4.3 Measure of Uncertainty Resolution Based on the Project Balance with the Expected Gain Confidence Limit Criterion

It is shown in Section 4.2 that the use of the coefficient of variation as a measure of uncertainty resolution has limitations so that it does not properly reveal how uncertainty about the project is reduced as its future outcomes are realized. The most significant weakness of the CV_t is the fact that it uses the standard deviation in its numerator and it therefore does not distinguish between variations on the up and the down side. In particular, since the CV_t is a ratio index the absolute magnitude of uncertainty is not properly revealed.

In order to appreciate the advantage of the expected gain confidence limit criterion (EGCL), the basic concept of the EGCL is formally discussed in Section 4.3.1. Since the EGCL recognizes the absolute magnitude of variability, it is in this spirit that a measure of uncertainty resolution for ANB and APB based on the EGCL is presented in Section 4.3.2, and compared with a measure of uncertainty resolution based on the CV_t using project balance in Section 4.3.3, respectively.

In this presentation, the decision maker is assumed to be risk averse. As discussed in Section 3.1.2.2, the decision maker who is risk averse will prefer a project whose ANB is smaller or whose APB is larger if other things are equal. Therefore, in terms of variability, the decision maker will be concerned more about the variability in down side for the APB and the variability in up side for the ANB. Accordingly, two different schemes of expected gain confidence limit criterion are utilized in this study to place confidence limits in the measurement of variabilities of the statistics (ANB and APB).

4.3.1 Expected Gain Confidence Limit Criterion

Suppose a decision maker wishes to select the better of two mutually exclusive projects, A and B, each of which has a payoff that can be represented as a random variable governed by a certain probability distribution. Suppose that $E_A \geq E_B$ (Mean) and $\sigma_A^2 \leq \sigma_B^2$ (Variance), with at least one of the inequalities being strict, then the rule of Markowitz [62] indicates that Project A would be preferred. In this case, it may well be said that Project B is dominated by Project A. On the contrary, if $E_A > E_B$ and

$\sigma_A^2 > \sigma_A^2$, then it cannot be determined whether the decision maker will select Project A or B without further information about his attitude toward risk. However, it is possible that the return of Project A is so much higher than Project B that under almost any circumstances it returns more than Project B, even though $\sigma_A^2 > \sigma_B^2$. Baumol [3] presents this idea by specifying some additional information represented by δ, a measure of risk aversion of the individual.

For each proposal i, the lower confidence limit, $L_i = E_i - \delta\sigma_i$ is computed. Baumol argues that Project A is preferable to Project B if $E_A \geq E_B$ and $L_A = E_A - \delta\sigma_A \geq L_B = E_B - \delta\sigma_B$, for a predetermined value of δ with at least one of the inequalities being strict. The value δ specified by the decision maker may be viewed as an index of his attitude toward unfavorable outcomes. Thus, the more conservative the decision maker, the larger will be the value assigned to δ and the lower will be the possible outcomes which are considered acceptable. If $\delta = 0$, then he is simply risk-indifferent and selects on the basis of expected returns only. In other words, δ represents the rate of trade-off between reduction (increase) in expected value and reduction (increase) in standard deviation ($\delta = \partial E/ \partial\sigma$). This implies that the larger the value of δ in $E_i - \delta\sigma_i$, the larger the increase in E_i required to compensate for a given increase in σ_i. Eventually, as δ goes to infinity, this rule becomes the same as the Markowitz rule [3]. Thus, when δ is very large, then the $\delta\sigma_i$ term becomes the dominating factor in $E_i - \delta\sigma_i$, and so the behavior of L_i will be almost perfectly correlated with that of σ_i.

To employ the method, the decision maker must first express his risk preferences by choosing a specific value for δ. When the return on the investment is normally distributed, then there is only about 16% probability that the realized return will ever fall below $E_i - (1)\sigma_i$, only a 2% probability that it will fall below than $E_i - 2\sigma_i$, no more than a 0.1% probability that it will fall below $E_i - 3\sigma_i$, etc. Thus, δ is equivalent to the standard deviation ($Z = \frac{X-E}{\sigma}$), for the normal distribution with $E = 0, \sigma^2 = 1$. But even where the distribution is not normal but unimodular, Chebyshev's inequality can be utilized to assigned quantitative evaluation of the likelihood that observations will fall beyond $E_i \pm \delta\,\sigma_i$.

4.3.2 The Use of the Expected Gain Confidence Limit Criterion in the Measure of Uncertainty Resolution

In this section, a time-dependent measure of uncertainty resolution based on the project balance with incorporation of the EGCL concept is presented. Two different schemes of resolution index are generated: one is for the ANB and the other for the APB.

4.3.2.1 Area of Negative Balance [ANB]. The resolution index for the area of negative project balance is defined as

$$EGCL[ANB]_t = E[ANB]_t + \delta\sigma[ANB]_t \tag{4-2}$$

where $EGCL[ANB]_t$ = resolution index for negative project balance at time t

$E[ANB]_t$ = conditional expected negative project balance at time t

δ = coefficient of risk aversion

$\sigma[ANB]_t$ = conditional standard deviation about the expected negative project balance at time t.

Here EGCL $[ANB]_t$ takes an upper confidence limit rather than a lower confidence limit for a risk-averse decision maker. Thus, if a project recovers its initial investment in a shorter time or a larger amount than expected, this deviation from the expected value cannot be regarded as an unfavorable outcome to the risk-averse individual. Instead, if the project takes a longer time or a smaller amount than expected, it would be an unfavorable outcome to the decision maker. Graphically, the shaded area in Figure 4-15 represents prob ($[ANB]_t \leq$ EGCL $[ANB]_t$), that is, the probability that the ANB at time t is greater than the tolerable limit EGCL $[ANB]_t$.

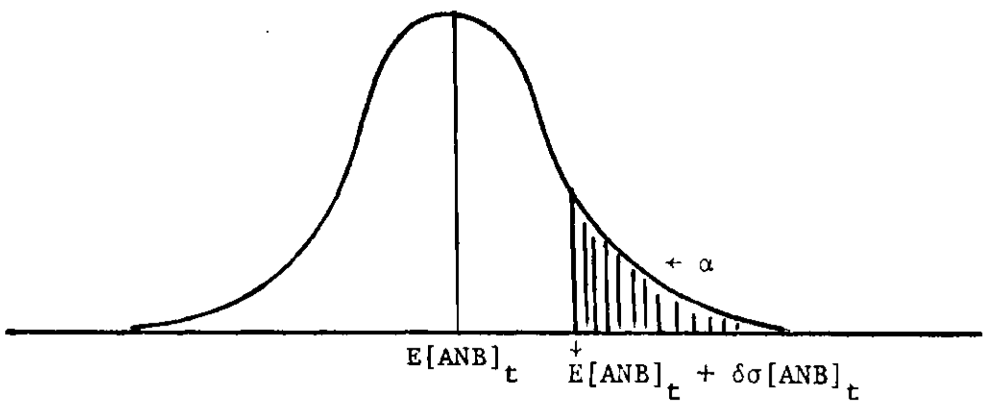


Figure 4-15. Upper Confidence Limit for $[ANB]_t$

If a random variable of return for the project is normally distributed and the decision maker is able to assign an upper bound on maximum tolerable probability α, then the following relationship holds between α and δ:

$$\text{Prob } ([ANB]_t \leq \text{EGCL}[ANB]_t) \leq \alpha$$

$$\text{Since} \qquad Z = \frac{EGCL[ANB]_t - E[ANB]_t}{\sigma[ANB]_t} = \frac{\delta\sigma[ANB]_t}{\sigma[ANB]_t} = \delta$$

In terms of the standard normal variate, δ can be expressed as probability $(Z < \delta)$, such that if α is given, then δ can be computed directly from the standard normal table.

If the distribution is not normal but is unimodular, then using Chebyshev's inequality, the relationship between α and δ may be expressed as:

$$\text{Prob } (E[ANB]_t - \delta\sigma[ANB]_t \leq [ANB]_t \leq E[ANB]_t + \delta\sigma[ANB]_t) \geq 1 - \frac{1}{\delta^2}$$

Graphically, the probability of observing a value $[ANB]_t \in T$ is given by $P(T) \geq 1 - \frac{1}{\delta^2}$

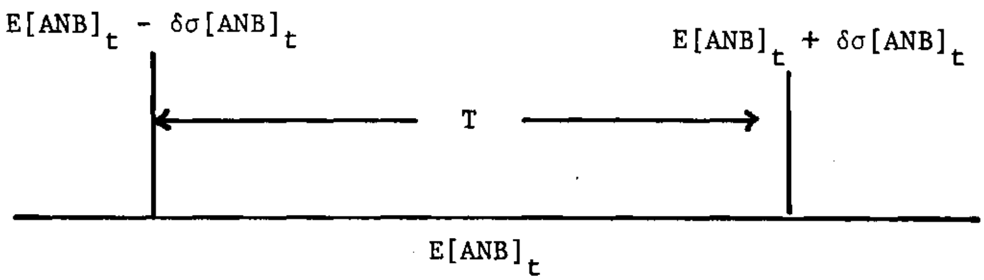


Figure 4-16. Illustration of Chebyshev's Inequality

Stated differently, for any distribution with given mean and variance, the probability of observations <u>outside</u> the range $E \pm \delta\sigma$ will be no greater than $1/\delta^2$. Then, for a given α, a decision maker would assign δ value with

$$\delta = (\frac{1}{\alpha})^{\frac{1}{2}}.$$

4.3.2.2 The Area of Positive Balance [APB]. The resolution index for the APB based on the expected gain confidence is defined as

$$EGCL[APB]_t = E[APB]_t - \delta\sigma[APB]_t \tag{4-3}$$

where $EGCL[APB]_t$ = resolution index for positive project balance at time t

$E[APB]_t$ = conditional expected positive project balance at time t

δ = coefficient of risk aversion

$\sigma[APB]_t$ = conditional standard deviation about the expected positive project balance at time t.

As contrasted with EGCL $[ANB]_t$, EGCL $[APB]_t$ takes a lower confidence limit for the decision maker who is risk averse. Here the realization of a random variable greater than E $[APB]_t$ is considered to be a favorable event. Thus, the lower limit would be placed somewhere below E $[APB]_t$, and it represents the point below which the realizations of $[APB]_t$ are undesirable as shown in Figure 4-17.

By adjusting the values of δ, the coefficient of risk aversion, the decision maker is able to reflect his personal preference for the resolution of possible losses and possible benefits over time. If the decision maker has a different view of the resolution of ANB and APB, then, of course, he may assign a different coefficient of risk aversion. The logical ground on which the value of δ should be based would be the same as the $[ANB]_t$. In this presentation, however, the same value of

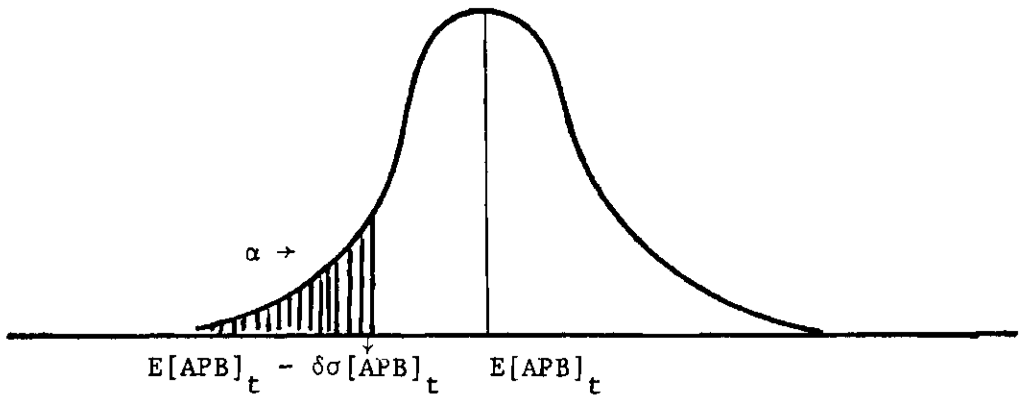


Figure 4-17. Lower Confidence Limit for $[APB]_t$

coefficient of risk aversion is assumed for both random variables $[ANB]_t$ and $[APB]_t$.

4.3.3 Comparison with the Coefficient of Variation Approach

To illustrate how the resolution index obtained from the expected gain confidence limit differs from the coefficient of variation approach, the same example given in Section 4.2.1 is used again here. Computations obtained from application of Equations 4-2 and 4-3 with varying values of δ (δ = 1, 2, 3 and 4), the resolution indexes for the ANB and APB are summarized in Table 4-7 and plotted in Figure 4-18. Table 4-7 indicates that Project A1 dominates Project A2 for a range of δ values from 1 to 4 with regard to ANB and APB. The resolution indexes in Table 4-7 reveal the true underlying cash flow stream discussed in Section 4.2.1. In particular, the resolution index obtained from the confidence limit represents both the magnitude of actual cash flows and the variability of cash flows simultaneously, whereas the coefficient of variation represents only the relative variability in cash flows.

However, as δ approaches infinity, the resolution indexes obtained

Table 4-7. Resolution Index Based on Expected Gain Confidence Limit for Projects A1 and A2

Project A1	ANB				APB			
t	δ=1	δ=2	δ=3	δ=4	δ=1	δ=2	δ=3	δ=4
0	151.2	186.4	221.5	256.6	14.1	-10.4	-34.8	-59.2
1	130.2	144.4	158.5	172.7	14.1	-10.4	-34.8	-59.2
2	124.5	132.9	141.3	149.8	15.4	- 7.6	-30.7	-53.7
3	116.1	116.1	116.1	116.1	38.5	38.5	38.5	38.5

Project A2	ANB				APB			
t	δ=1	δ=2	δ=3	δ=4	δ=1	δ=2	δ=3	δ=4
0	260.1	274.2	288.3	302.4	- 1.2	-26.3	-51.4	-76.5
1	259.2	272.5	285.7	298.9	.2	-23.7	-47.5	-71.3
2	254.4	262.8	271.3	279.7	.9	-22.2	-45.2	-68.3
3	246.0	246.0	246.0	246.0	24.0	24.0	24.0	24.0

from either criterion (EGCL or $\bar{CV}_t$) will not differ significantly from the other. In other words, when δ is very large, $\bar{CV}_t$ and $EGCL[ANB]_t$ (or $EGCL[APB]_t$) will indicate the same set of project preferences. Mathematically, the relationship between $EGCL[ANB]_t$ (or $EGCL[APB]_t$) and $\bar{CV}_t$ can be expressed as

$$
\begin{aligned}
EGCL[ANB]_t &= E[ANB]_t + \delta\sigma[ANB]_t \\
&= E[ANB]_t \left[1 + \delta \left(\frac{\sigma[ANB]_t}{E[ANB]_t}\right)\right] \\
&= E[ANB]_t \left[1 + \delta\bar{CV}_t\right], \quad \left(\text{since } \bar{CV}_t = \frac{\sigma[ANB]_t}{E[ANB]_t} \text{ for project balance}\right) \qquad (4\text{-}4)
\end{aligned}
$$

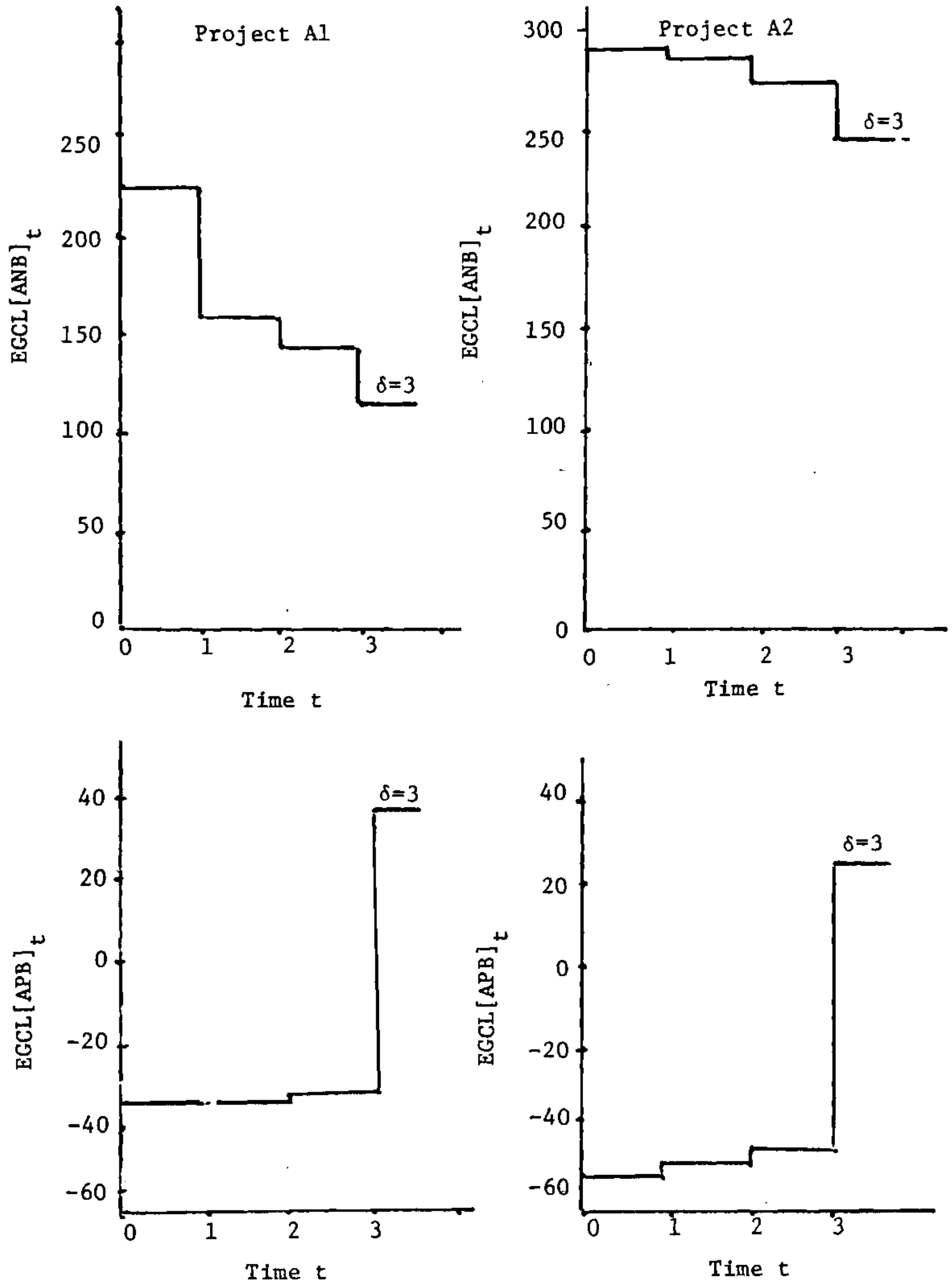


Figure 4-18. Uncertainty Resolution Patterns of $[ANB]_t$ and $[APB]_t$ for Projects A1 and A2

When δ is large, the $\delta C\bar{V}_t$ term becomes the predominant term in Equation 4-4; thus, the behavior of $EGCL[ANB]_t$ will be almost perfectly correlated with that of $E[ANB]_t C\bar{V}_t$.

From the foregoing discussion, it is evident that the resolution index based on project balance with incorporation of the concept of the EGCL ought to be used as a time-dependent measure of uncertainty resolution. Therefore, the measure of uncertainty resolution adopted in this study is $EGCL[ANB]_t$ for the ANB and $EGCL[APB]_t$ for the APB.

CHAPTER V

DECISION CRITERION FOR SEQUENTIAL CAPITAL BUDGETING PROBLEMS

In Chapter III, the project balance as a time-dependent measure of investment worth was proposed and its uniqueness as compared to the traditional measures of investment worth was discussed. Since the project balance provides more information about the nature of cash flows with respect to time, the use of the project balance as a basis to measure uncertainty resolution with the expected gain confidence limit approach was presented in Chapter IV.

This chapter incorporates the concept of uncertainty resolution which was developed in Chapter IV in a systematic manner into a framework for investment decisions. To utilize information about the resolution of uncertainty over time, a decision criterion called the Project Balance Criterion is developed as a method to assess the effect of investment projects that have probabilistic outcomes.

Since most actual investment situations require the repeated consideration of projects over time, it is important to understand how the ideas previously developed can be applied on a sequential periodic basis. Therefore, this chapter begins with a discussion of the general investment decision process and the objectives of the firm where the firm makes capital allocations on a regular periodic basis. In Section 5.2, the development of the project balance criterion and its application to a sequential decision process is described.

One of the primary purposes of this study is to compare three popular decision criteria with the project balance criterion for the type of investment decision process defined in Section 5.1. Therefore, in Section 5.3 these decision criteria are described in detail. In addition to comparing these three criteria with the project balance criterion, the assumption of having perfect information is also introduced. This assumption allows the comparison of the project balance criterion with well known deterministic models. Thus, comparison among methods requiring perfect information and methods designed for probabilistic information can be presented.

5.1 General Investment Situations

The primary objective of this study is to investigate the problem of making capital allocations on a regular periodic basis when there is a lack of complete information as to the events that might occur in future decision periods. In particular, the investment situation is defined as one in which the occurrence, timing, and characteristics of future investment opportunities cannot be predicted with certainty. This type of investment setting is of particular interest because decision making under imperfect information fits the situation most businessmen face in reality [8].

In Section 5.1.1, the general investment situations which postulate the data-generating process for future investment opportunities, as well as the characteristics of those investments, are described. In Section 5.1.2, the objectives of the firm are defined for the investment situations described in Section 5.1.1.

5.1.1 Description of the Decision Process

Consider a hypothetical firm which is faced with a number of potential investment opportunities to evaluate in each operating period. The projects undertaken provide a set of possible cash flows over several operating periods. The future cash flows generated by these investments are not precisely known and they can be represented by the probability tree described in Section 4.1. In other words, at the time of decision precise knowledge of the future realizations of cash flows for those proposals being considered is not available. However, at each decision point, the size of the initial investment and the life of each proposal are known with certainty.

The firm lacks complete information about the investment opportunities arising in the subsequent decision periods. If a set of investment proposals to be considered is identified during each decision period, the firm then must allocate the funds available for that period to the set of proposals according to some decision criterion. Once the decision is made, the funds are invested in the selected proposals, and the rejected proposals are discarded from further consideration for future periods. A new set of proposals becomes available during the following period, and the firm must again make an allocation decision at the end of that period. This process continues over the planning horizon time.

It is not known precisely what the budget will be in the future periods because the budget at any future decision point is determined by the net cash received from investments made in prior periods. Because the firm finances all of its investments through the reinvestment of internally generated funds, the option of outside borrowing is not considered in the study. In other words, the restriction that no external funds are allowed means that the firm cannot obtain additional funds in order to affect its decision. There are reasons for not allowing external funds in this study. One is that many firms as a matter of policy will not borrow funds to finance investment proposals. The other reason is that the inclusion of borrowing in the decision process does not require a decision criterion to be as selective and therefore the efficacy of the criteria considered here would not be thoroughly tested.

However, if the proposals selected at a decision time do not exhaust the budget, the funds that remain are invested in a highly liquid investment at some interest rate i_δ. This i_δ results from a highly liquid investment such as a bank account where the funds may be withdrawn at any time. Thus, the decision maker will have these invested funds plus interest available for investment at the next decision time.

5.1.2 Objectives of the Firm

Given this general setting of investment decision situations, the basic problem is to select from a set of available investment opportunities a subset which maximizes the total capital at the horizon time, subject to certain budget restrictions which must be satisfied by any feasible investment plan. In each decision period, the decision maker has to select a set of proposals based on a certain objective decision rule. When comparing sets of uncertain cash flow sequences over a multi-period investment horizon, each decision ought to consider information which reflects the effects of project variability on the ultimate accumulation of wealth. In this study the information utilized includes (1) the expected net present value; (2) the variance about this expected value; (3) uncertainty about changes in the project's ultimate worth as the project's costs and revenues are actually realized (uncertainty resolution, see Section 4.3).

The difficulty with establishing a single decision criterion which incorporates all the three factors is that difficult trade-offs are required. Management usually is able to tolerate greater variability about the project's worth if the expected profitability is great enough. Or, management might wish to select a current set of projects with a lower expected net present value, but a more rapid resolution of uncertainty in the project's negative balance in order to take advantage of investment opportunities which would arise in the future. Thus, the decision criterion to be devised must be one which allows for trade-offs among these three factors. Therefore, a decision criterion which explicitly incorporates the uncertainty resolution concept along with the mean-variance properties is developed in the following section.

5.2 Development of the Project Balance Criterion

In Chapter IV, an information framework for analyzing uncertainty resolution was developed. In particular, the index developed in Section 4.3 represents a time-dependent measure of uncertainty resolution that provides the management with information about the changes in uncertainty over time for the projects under consideration. Such information can be of particular use to a risk-averse decision maker who has a basic feeling for the general manner in which new investment opportunities may occur for his particular firm as time passes.

The investment situations described in Section 5.1 and the idea developed in Section 4.3 suggest a criterion for making investment decisions with incorporation of the concept of uncertainty resolution on a regular periodic basis. Thus, in this section the criterion, called the project balance criterion (PB criterion), is formally proposed, and its application to a sequential decision process is described. In order to appreciate the advantage of the PB criterion when decisions are made on a regular periodic basis, it is important to understand its logical derivation.

In Section 5.2.1, the specific decision problem in which the project balance criterion is to be developed is described in detail. In Section 5.2.2, a single index is devised as an operational decision rule to seek a practical trade-off among the three major investment factors outlined in Section 5.1.2.

5.2.1 Decision Problem

Consider the investment situation that is described in Section 5.1.1 where the decision maker has to make capital investment decisions on a regular periodic basis with the objective of maximizing his future wealth at the horizon time H. In particular, the decision maker is aware of all the proposals generated during a particular time interval at the time of decision, but no precise estimate about the future realization of the proposal's cash flows can be made. Furthermore, the decision maker has no precise knowledge of the investment proposals that will be submitted for consideration in future periods.

More specifically, assume that the decision maker has to make investment decisions on a regular basis with the following investment situations:

1. The set of proposals under consideration, which will be referred to as the schedule of investment proposals (SIP), are known at decision

points in time, but the proposals' future cash flows contained in the SIP at that time are not known with certainty. However, the probabilities of occurrence of various outcomes for these cash flows can be estimated and represented by a probability tree such as described in Section 4.1.

2. The SIP cannot be identified for decision periods in advance. That is, the SIP can be completely identified <u>only</u> at the time of each decision period.
3. The amount B_0 is budgeted for investment at the decision time $t = 0$. For $t \geq 1$, the amount B_t over the subsequent investment period is to be conditional upon the acceptance or rejection of projects available in the previous decision periods.
4. The discount rate for project evaluation (MARR) and the rate i_δ which can be carried on funds not invested are known in advance. The planning horizon time span, H, also is assumed to be known.

For these investment situations, the decision maker is faced with selecting a set of projects at each decision period which would result in the maximization of his capital at the horizon time. Since the future receipts of the projects are not deterministic, an aware decision maker must be concerned about the variability of the resultant cash flows in addition to the expected profitability of the proposal. In other words, questions of risk preference require the inclusion of measures other than expected equivalent profit in any sound decision criterion.

5.2.2 Use of Project Balance Pattern Parameters in the Project Balance Criterion

As developed in Sections 3.1 and 3.2, the project balance pattern provides four different elements of information regarding the desirability of a proposal. They are the area of negative project balance (ANB), the time for the project to recover its initial investment (Q), the area of positive balance (APB), and the terminal profitability ($S_N(i)$) of the proposal (see Figure 3-7). However, as a basis for measuring uncertainty resolution in Chapter IV, only two of these elements (ANB, APB) were utilized to generate two different resolution indexes for the same proposal. One is for the negative project balance and the other for the positive project balance (see Section 4.3). In particular, Q and $S_N(i)$ are not utilized as a basis for measuring the rate at which uncertainty about the realizations of cash flow changes through time. The reasons for selecting on these two elements are discussed in Section 3.2.

Since both resolution indexes, $EGCL[ANB]_t$ and $EGCL[APB]_t$, represent time-dependent measures of uncertainty resolution, the use of information derived from these resolution indexes in the development of the project balance criterion is discussed. In particular, Section 5.2.2.1 describes the conceptual advantage using $EGCL[ANB]_t$ over $EGCL[APB]_t$ in the development of the project balance criterion, where investment decisions are made on a regular periodic basis. In fact, information generated from $EGCL[APB]_t$ is not utilized in the project balance criterion, with the reasons being discussed in Section 5.2.2.1. However, in Section 5.2.2.2, the possible use of information derived from $EGCL[APB]_t$ is discussed in a different capital budgeting context.

5.2.2.1 The Use of Information about Uncertainty Resolution Provided from the Area of Negative Project Balance. In general, as the expected discounted payoff period approaches the investment life, the value of information about the resolution pattern for the positive project balance would tend to decrease. In particular, when liquidity preference can be viewed as behavior regarding risk [93], the risk-averse decision maker would tend to place more value on the resolution of uncertainty for negative project balance. Following the definitions of Tobin [93], project liquidity is defined as follows: when the project has a relatively short time period to recover its initial investment, it is said the investments made are relatively liquid. In this sense, liquidity preference as behavior for a risk-averse decision maker can be viewed as meaning that the investor wishes to assure himself of being restored to his initial position within a short span of time in order to be able to take advantage of additional, perhaps better investment possibilities that may come along [104]. If this is the case, he may be less interested in knowing the way the terminal profit accumulates once the initial investment is fully recovered. Therefore, after he recovers the initial investment, his prime concern should be terminal profit.

In a context of sequential capital budgeting problems, the primary emphasis of this study is centered on incorporating the information provided by the area of negative project balance into the decision criterion. This is because an early resolution of negative project balance would make funds available for attractive investment opportunities which would arise in the subsequent decision periods.

5.2.2.2 The Possible Use of Information Provided by Area of Positive Project Balance. Even though information generated from the APB of the proposals is not directly used in the development of the PB criterion, this information may have some usefulness in aiding the decision maker in his attempts to discontinue some marginal projects. When the abandonment decision is considered in the decision process, for projects whose expected net present values are somewhat smaller and uncertainty about the realizations of future cash flows persists longer than usual, it would be good judgment to consider them for early retirement because more could be earned from their sale for salvage. That is, in capital rationing situations, abandonment is possible whenever the need for cash arises in order to invest in a more lucrative project. Schwab and Lusztig [85] summarize the usefulness of the abandonment consideration in investment decisions such as:

> The economic desirability of an investment project is not only determined by uncertainties in future cash flows which may lead to premature abandonment, but that it also depends on the future availability of new and better investment alternatives which may lead to premature abandonment even when net present values are greater than the abandonment values over all relevant time periods.

Thus, the information generated from the uncertainty resolution for the positive project balance ($EGCL[APB]_t$, or the resolution index of the profit accumulation pattern), may provide the decision maker with a sort of tracking signal to examine the possibility of abandonment (see Section 3.1.2.5).

The difficulty of considering the abandonment decision in capital budgeting problems is that it largely depends on the salvage value available at each point in time, which is difficult to estimate at best, particularly for new projects. Since the abandonment decision is not a primary concern of the decision criterion to be developed in this study, no additional consideration is given to incorporation of the abandonment decision into the current criterion.

5.2.3 Trade-offs Among Three Major Investment Factors

As pointed out in Section 5.1.2, when comparing sets of uncertain cash flow sequences over a multi-period investment horizon, each decision ought to consider information which reflects the effects of project variability on the ultimate accumulation of wealth. Through the discussion in Section 5.2.2, the following three elements are defined as the major investment factors which affect investment decisions in this study.

1. The expected terminal profitability (profitability)
2. The variability about this expected terminal profitability (variability)
3. Flexibility in future investment activity (Flexibility)

The project balance criterion seeks a trade-off among the three major factors defined above. The approach taken in this study is as follows: the terminal profitability is measured by the expected net present value of the project; the variability in the net present value is expressed in terms of standard deviation from the expected net present value; and liquidity preference is measured by the uncertainty resolution of negative project balance through time ($EGCL[ANB]_t$). Thus, graphically, the trade-off among these factors should reside inside a triangular space whose three corners are shared by the three investment factors.

To devise an operational and practical decision rule which indicates whether a project should be undertaken or should be rejected, a single index which brings these major parameters together is necessary. Therefore, subsequent sections discuss what trade-off among the parameters should result in a single index.

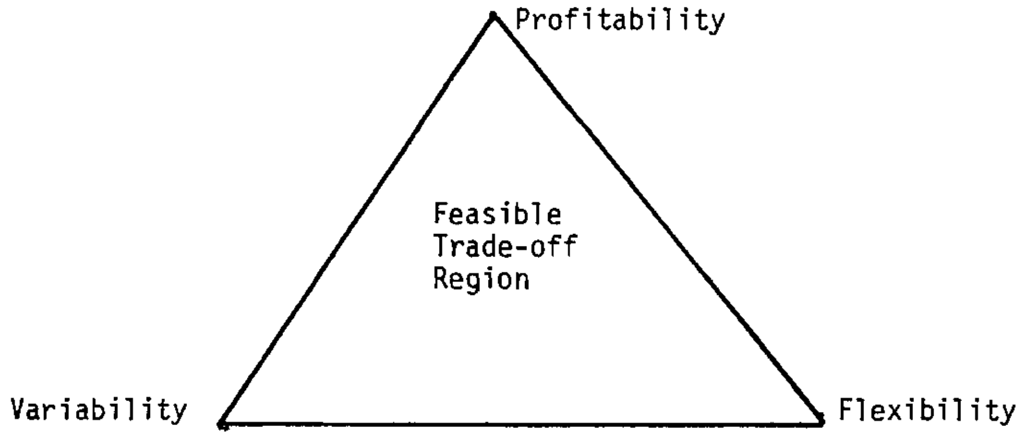


Figure 5-1. Trade-offs among Profitability, Variability and Flexibility

5.2.3.1 Trade-Off between Profitability and Variability. The first task is to establish a trade-off between the expected net present worth of the return for the proposal (E) and the variability of this return (σ). The approach taken in this study is a lower confidence limit concept discussed in Roy [83] and Baumol [3]. That is, the floor on project returns (lower confidence profit limit) is represented by Equation 5-1.

$$V_i = E_i - k\sigma_i \tag{5-1}$$

where V_i = lower confidence profit limit of i^{th} investment opportunity

E_i = expected net present worth for the i^{th} investment opportunity

k = coefficient of risk aversion

σ_i = standard deviation of the probability distribution of the net present worth i^{th} investment opportunity

Equation 5-1 summarizes the two dimensions, expected value and variability, by which an uncertain prospect is considered to be characterized with a single index on which accept-reject decisions can be made. Here, k can be regarded as the rate of trade-off between reduction in expected value and for reduction in variability. Conceptually, it can be said that increases in an investment's expected value E_i increase its desirability, while increases in the dispersion of possible outcomes tend to reduce an investment's desirability. These relationships may be expressed symbolically as

$$\partial V_i/\partial E_i < 0,\ \partial V_i/\partial \sigma_i < 0$$

and $$d\sigma_i/dE_i = (\partial V_i/\partial E_i)(\partial \sigma_i/\partial V_i) = k^{-1}$$

Figure 5-2 graphically depicts the trade-off between E_i and σ_i.

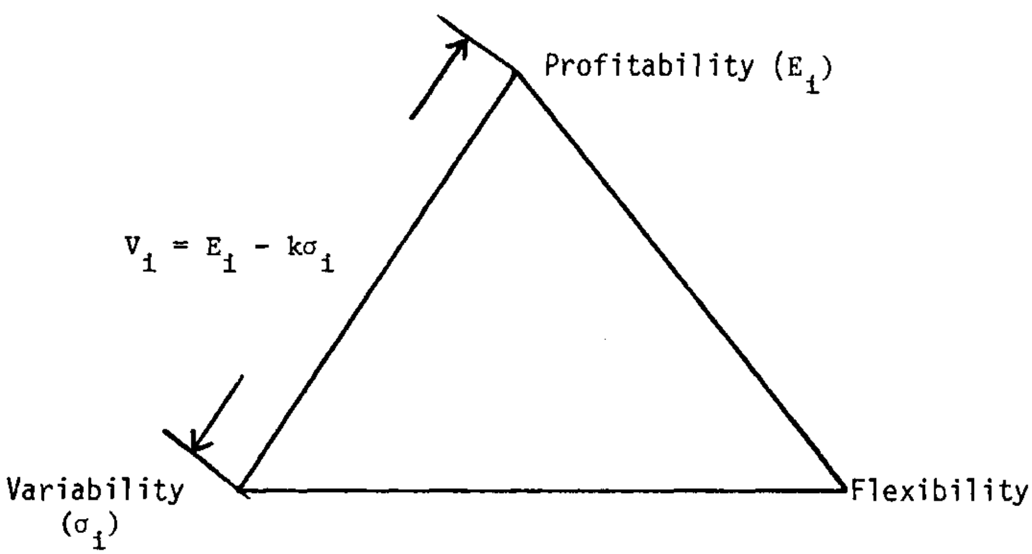


Figure 5-2. Trade-off between Profitability and Variability

5.2.3.2 Trade-off between Lower Confidence Profit Limit (V_i) and Uncertainty Resolution ($EGCL[ANB]_t$). Once a trade-off is achieved between the profitability and the variability of the proposal, the next step involves a trade-off between uncertainty resolution and V_i. However, the expected pattern of uncertainty resolution through time is not a single dimension itself; to translate such a pattern into an operational decision criterion requires a single numerical index. The reason why a single index is desirable is that to be at all realistic a decision rule which considers the shape of whole pattern of uncertainty resolution would necessarily be quite complex and unwieldy.

There would be two possible ways to summarize information contained in the expected pattern of uncertainty resolution by a single index. One approach is to reduce the multi-period concept of uncertainty resolution into a single index, represented by the area under the uncertainty resolution of negative project balance pattern. Another possibility would use an average rate of decline $EGCL[ANB]_t$ for all periods as used by Van Horne [95]. However, this approach requires a ratio index such that the absolute magnitude of uncertainty cannot be revealed appropriately. Thus, for this study the total uncertainty resolution for the negative project balance is measured by the area under the pattern of uncertainty resolution $[ANB]_i$, as shown in Figure 5-3.

$$[ANB]_i = \sum_{t=0}^{N} EGCL[ANB]_{it} \qquad (5\text{-}2)$$

However, it must be realized that this simplification results in a certain

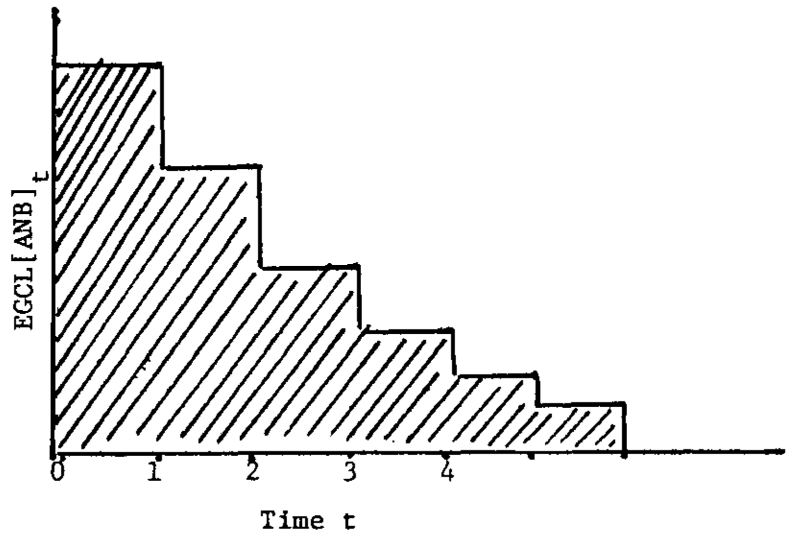


Figure 5-3. Area under the Uncertainty Resolution Pattern

loss of information. As an example, suppose that two alternative current investment opportunities generate resolution indexes (EGCL[ANB]$_t$) as shown in Figure 5-4. Area transformation indicates that both proposals are indifferent in terms of uncertainty resolution. However, although both of

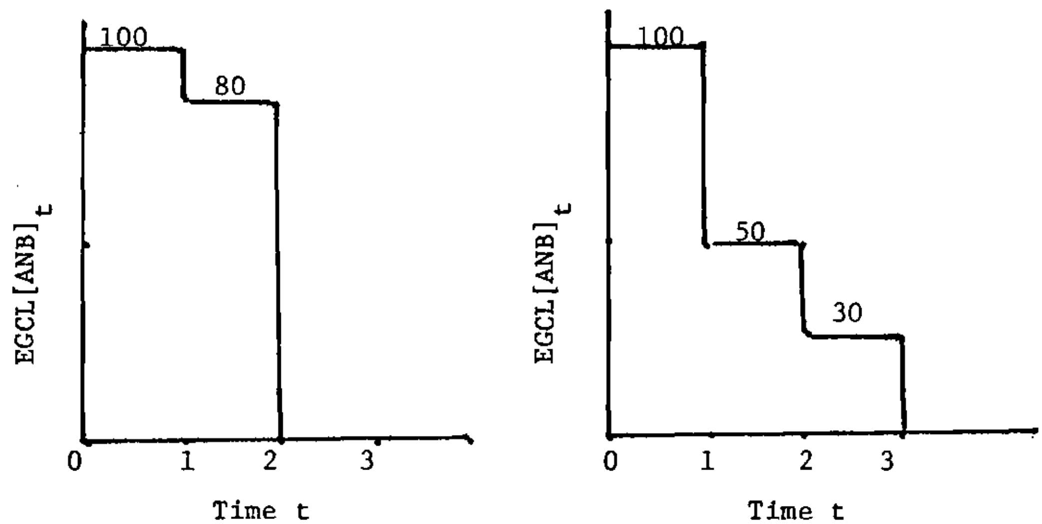


Figure 5-4. Alternative Patterns of Proposal's Uncertainty Resolution

the investments contain the same initial amount of uncertainty, Proposal H1 completely resolves the uncertainty in two years, whereas this is not true of Proposal H2. This type of information would be lost by summarizing the uncertainty resolution characteristics for negative project balance as the area under the pattern of uncertainty resolution.

To eliminate the effect of the size of the initial investment, it is necessary to normalize ANB by dividing throughout by the initial investment cost C_i.

$$[ETIP]_i = [ANB]_i/C_i \tag{5-3}$$

This normalized area is equivalent to the length of a side of a rectangle whose area is the same as $[ANB]_i$ and whose other side corresponds to the initial investment cost. This relationship is depicted in Figure 5-5.

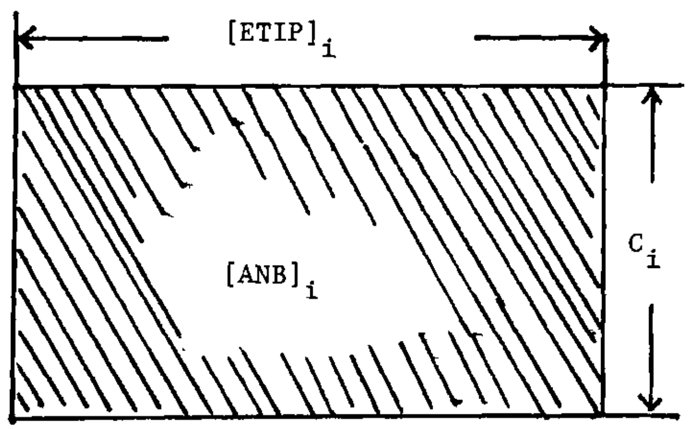


Figure 5-5. Expected Total Investment Period $[ETIP]_i$

The parameter $[ETIP]_i$ which stands for <u>expected total investment period</u>

can be viewed as the equivalent period during which the total project balance (debt) would be outstanding. Now the final task is to express a trade-off between $[ETIP]_i$ and V_i. Since V_i represents dollar amount and $[ETIP]_i$ has a time dimension, $V_i/[ETIP]_i$ would represent the certainty equivalent amount (lower confidence profit limit) per investment period from the i^{th} investment opportunity.

$$Z_i = V_i/[ETIP] \qquad (5\text{-}4)$$

$$= [E_i - \delta\sigma_i]/[\sum_t^N EGCL(AND)_{it}/C_i]$$

The graphical representation of the trade-off among the three investment factors can be depicted as shown in Figure 5-6. Conceptually, the final trade-off point Z_i should reside inside a triangular space whose three corners are shared by the three investment factors.

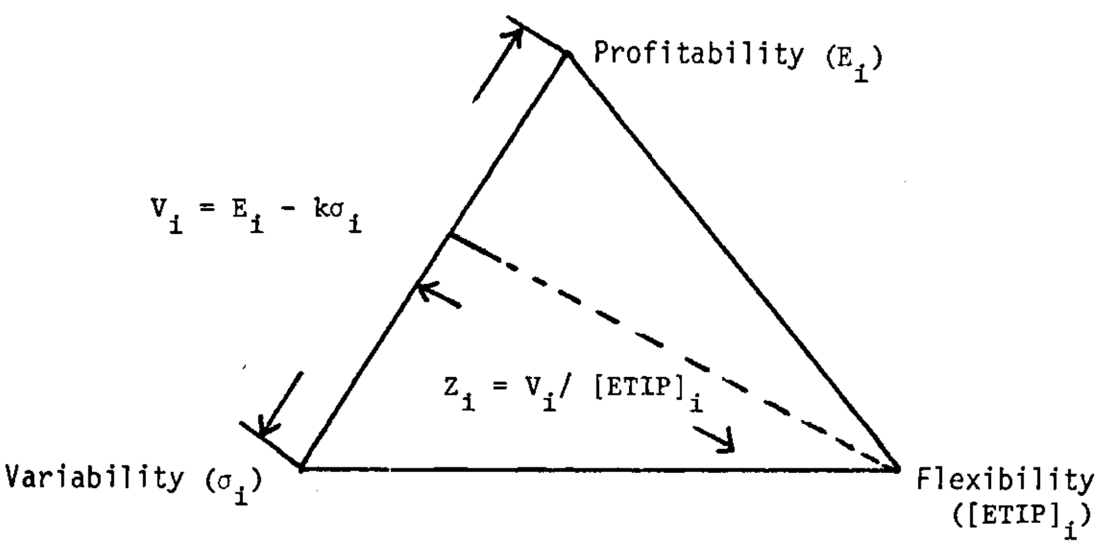


Figure 5-6. Final Trade-off among Critical Investment Factors

5.2.4 Project Balance Criterion

The problem of selecting the combination of investments that maximizes the certainty equivalent amount per investment period for a particular set X of investment proposals is equivalent to solving the following linear integer programming problem at each decision period:

$$\text{Model I:} \quad \text{Maximize } Z = \sum_{i}^{n} Z_i X_i$$

$$\sum_{i} C_i \leq B$$

$$X_i = (0,1) \text{ integer}$$

where

B = budget limit

C_i = initial investment of proposal i

$$X_i = \begin{cases} 0, & \text{if proposal } i \text{ is not selected} \\ 1, & \text{if proposal } i \text{ is selected} \end{cases}$$

$Z_i = V_i/[\text{ETIP}]$

5.2.4.1 Consideration of Interrelated Proposals. Since the project balance criterion is formulated as a linear integer programming problem, it is easy to accommodate proposals which have dependency relationships (e.g., mutually exclusive or contingent proposals). A set of mutually exclusive opportunities is handled simply by adding the following restrictions for each set of mutually exclusive proposals:

$$\sum_{i \varepsilon S} X_i \leq 1$$

where the summation is over a set S of mutually exclusive projects contained in the decision maker's SIP. This restriction ensures that at most only one proposal of the set S will be accepted.

To handle contingency relationships, another constraint must be added in the following form:

$$X_\ell \leq X_m$$

where acceptance of proposal ℓ is assumed to be conditional upon acceptance of proposal m (see Weingartner [101]).

5.2.4.2 Consideration of Covariances among Proposals. As shown in Equation 5-4, the project balance criterion postulates that the variance of the returns from a single project is an appropriate measure of risk. In other words, the criterion is designed to evaluate one project at a time rather than evaluating a portfolio of projects. However, when it is desirable to evaluate risk to the firm as a whole, theoretically the risk of a proposal should be evaluated with respect to the risk characteristics of the other proposals. It may be judged in relation to its marginal additions of risk to the firm as a whole. This implies that covariances among the proposals in the combination must be recognized to determine the overall risk. Should these interdependencies be recognized, the project balance criterion would be expressed as

$$\text{Model II:}\quad \text{Maximize } Z = \frac{\sum_i E_i X_i - k\left(\sum_i \sum_j X_i \sigma_{ij} X_j\right)^{\frac{1}{2}}}{\left[\sum_i \{\sum_t E(ANB)_{it} X_i + \delta \sum_t \left(\sum_i \sum_j X_i \sigma'_{ijt} X_j\right)^{\frac{1}{2}}\}\right] / \sum_i C_i}$$

subject to $\sum_i C_i X_i \leq B$

$X_i = (0,1)$ integer

where σ'_{ijt} = covariance of ANB_t between proposal i and proposal j

$E(ANB)_{it}$ = expected ANB of Proposal i at time t

Since the objective function is expressed in a nonlinear form, it is difficult to solve the model given above with the existing nonlinear programming techniques. For a small number of proposals, a complete enumeration technique would be possible to obtain the optimal solution for the problem. This is done simply by excluding combinations of projects that violate the budget constraint from the feasible set. Another possible, but rather complicated alternative is to use first integer quadratic programming to obtain the optimal solution set which maximizes V

$$MAX\ V = MAX[\sum_i E_i X_i - k(\sum_{ij}\sum X_i \sigma_{ij} X_j)^{\frac{1}{2}}]$$

with the set of constraints defined in Model II. Let this solution set be S_1. Then, utilizing the same integer quadratic programming routine, solve the problem which <u>minimizes</u> $[ETIP]_z$ with the same set of constraints given in Model II. Let the solution set obtained from the latter be S_2. If $S_1 = S_2$, stop because the optimal solution has been found. If $S_1 \neq S_2$, the optimal solution can be obtained through exchange operations between the solution variables S_1 and S_2, and by computing $V/[EITP]_z$ which is maximum (see also Peterson and Laughhunn [72]).

Using an adaptation of Sharpe's "diagonal model (index model)" as a simplification of Model II, one can simplify the computations for obtaining optimal combination of projects (see Sharpe [86]). The use of the index model assumes that the returns on proposals are related only through a relationship with some common factor, e.g., an index of general market activity such that the index model allows only a restricted covariance between the eligible projects. Although an extension of Sharpe's work to Model II simplifies the computations, the use of the index model requires prediction of some additional parameters (see Sharpe [86]) which are rather difficult to obtain [102]. Because for practical situations it is difficult to find the covariances among the proposals in a meaningful way, covariance is not considered in this study [11, 16, 52].

5.3 Decision Criteria to Be Compared with the Project Balance Criterion for Sequential Decision Process

Several decision criteria have been proposed for evaluating the profitability of investment projects under risk (see Section 2.2.3). Most decision techniques consider risk by utilizing probability distributions of possible investment outcomes [43, 100, 38, 70]. Another widely discussed approach to evaluating investment proposals under risk is based on utility theory [31, 45, 12]. Thus, three of the most widely suggested decision criteria under risk in the capital budgeting literature are selected to compare with the PB criterion for the type of investment decision process described in Section 5.1. They are the expected present worth criterion, the mean variance criterion, and the expected utility criterion.

In addition to comparison of these three criteria with the PB criterion, decision making with perfect information is introduced to compare the effectiveness of the PB criterion in relation to the three decision criteria. Decision making with perfect information is defined as the investment situation where the decision maker has complete knowledge about the future investment opportunities and the realizations of their future cash flows. Two different levels of knowledge are examined. The first level assumes that the decision maker has complete information at the time of decision about those investment proposals that are currently being considered. The second level assumes that complete knowledge exists regarding all the proposals being considered throughout the study period. The second level certainly requires superior insight to that needed for the first level.

One of the primary objectives of this study is to investigate the performance of different decision criteria in more realistic investment situations. Most of the articles describing or commenting on these criteria limit their arguments to the selection of a small number of projects at one decision time. Therefore, the application of these decision criteria along with the PB criterion for making regular periodic decisions requiring selections from a large number of proposals is also presented.

5.3.1 Expected Present Worth Maximization

Under the expected present worth criterion, the decision maker is assumed to be risk-indifferent such that his problem is to select the feasible solution vector $(x_1, x_2, \ldots, x_n)$ having the largest expected net present value without violating the budget constraints. The vector is obtainable by solving the zero-one integer programming problem.

Model III: Maximize $Z = \sum_i [E_i] X_i$

subject to $\sum_i (C_i) X_i \leq B$

$X_i = (0,1), \forall i$

The implicit assumption using formulation Model III is that the decision maker's utility function in z can be expressed in the form of

$$U(z) = a + bz$$

where a, b are constants.

By taking the expected value operator,

$$E(U(z)) = a + bE(z) \tag{5-5}$$

Thus, Equation 5-5 implies that the decision maker can maximize his expected utility using Model III if he has a linear utility function.

It is important to recognize that the present worth maximization criterion is a special case of the PB Criterion. In Equation 5-4, if all projects liquidate themselves within <u>one period</u> with certainty, then Equation 5-4 can be reduced to

$$Z_i = \frac{E_i - k\sigma_i}{[\Sigma EGCL(ANB)_{it}]/C_i} = E_i - k\sigma_i$$

that is,

$$[\sum_t EGCL(ANB)_{it}]/C_i \rightarrow 1$$

a zero-one integer quadratic programming formulation. The application of this model uses a project's net present value (NPV) at a risk-free discount rate with a budget constraint.

$$\text{Model V:}\quad \text{Maximize } E - \lambda\sigma^2 = \sum_i^n E\,X_i - \lambda\left(\sum_i \sum_j X_i \sigma_{ij} X_j\right)$$

$$\text{subject to}\quad \sum C_i X_i \leq B$$

$$X_i = (0,1)\ \forall\ i$$

where

E = expected NPV of the accepted set of projects

σ^2 = variance of NPV of the accepted set of projects

λ = coefficient of risk aversion

E_i = expected NPV of project i

σ_{ij} = covariance of NPV of project i and project j

$\sigma_{ii} = \sigma_i^2$ = variance of NPV of project i

C_i = initial investment cost of project i

B = budget amount

$$X_i = \begin{cases} 0, & \text{if project } i \text{ is rejected} \\ 1, & \text{if project } i \text{ is accepted} \end{cases}$$

Then, the solution to Model V when varying λ value will generate the efficient frontier indicating the achievable risk-return trade-offs, given the firm's available investment alternatives.

If all the investment projects are mutually independent, Model V can be rewritten (covariance = 0) as

Model VI: Maximize

$$E - \lambda\sigma^2 = \sum_i X_i E_i - \lambda(\sum_i \sigma_i^2)$$

$$= \sum_i (E_i - \lambda\sigma_i^2) X_i$$

subject to

$$\sum_i C_i X_i \leq B$$

$$X_i = (0,1), \ \forall\ i$$

which can be solved as a zero-one integer programming model.

5.3.3 Expected Utility Criterion

One of the most widely discussed approaches towards risk in the capital budgeting literature is the expected utility criterion. This approach, based on utility theory concepts incorporates utility preferences into the investment decision by use of a utility function. The first and most difficult requirement of this criterion is to derive and specify the utility function that is to be applied. Commonly, a cardinal measure of utility provides a basis for assigning preference orderings for uncertain prospects [99].

Once the decision maker's utility function is specified, the expected utility value of a particular investment can be computed by multiplying the utile value of a particular outcome times the probability of occurrence and adding together the products for all probabilities. When a set of projects are compared, the project providing the highest expected utility value would be the most preferred alternative under the expected utility criterion. Thus, if the utility theory approach to capital budgeting is employed, the decision maker would seek to maximize the expected utility value of investment proposals under consideration, given a set of budget constraints.

Theoretically, in order to select the best combination of the investments under consideration, it is necessary to derive the probability distribution of the present value for each feasible combination that maximizes the expected value of a utility function. In fact, the number of feasible utility functions is practically infinite. Therefore, it is recognized at the outset that drawing some general conclusions about the effectiveness of the expected utility maximization as a decision criterion by testing a particular utility function is not reasonable. It also is impossible to test all the cases of utility function. Furthermore, the computations required to generate such probability distributions of the present value for each feasible combination are considerable. In view of these facts, an approximate method is utilized in this study. Some accuracy is sacrificed, but the saving in complexity is significant. It must be recognized that the purpose of the inclusion of this criterion in the study is not to draw general conclusions regarding the effectiveness of the criterion as compared to other criteria, but to provide the reader with some inference about the criterion.

5.3.3.1 Diminishing Marginal Utility and Risk Aversion. In order to understand the fundamental properties of the utility function for a risk-averse decision maker, it is necessary to examine how the risk element of the project is reflected in the utility function. As shown by Von Neumann and Morgenstern [99], risk-averse behavior will result if the decision maker has a diminishing marginal utility of returns. In other words, each additional dollar gives less utility or satisfaction than all preceding investments of the same size. Such a utility function is shown in Figure 5-7.

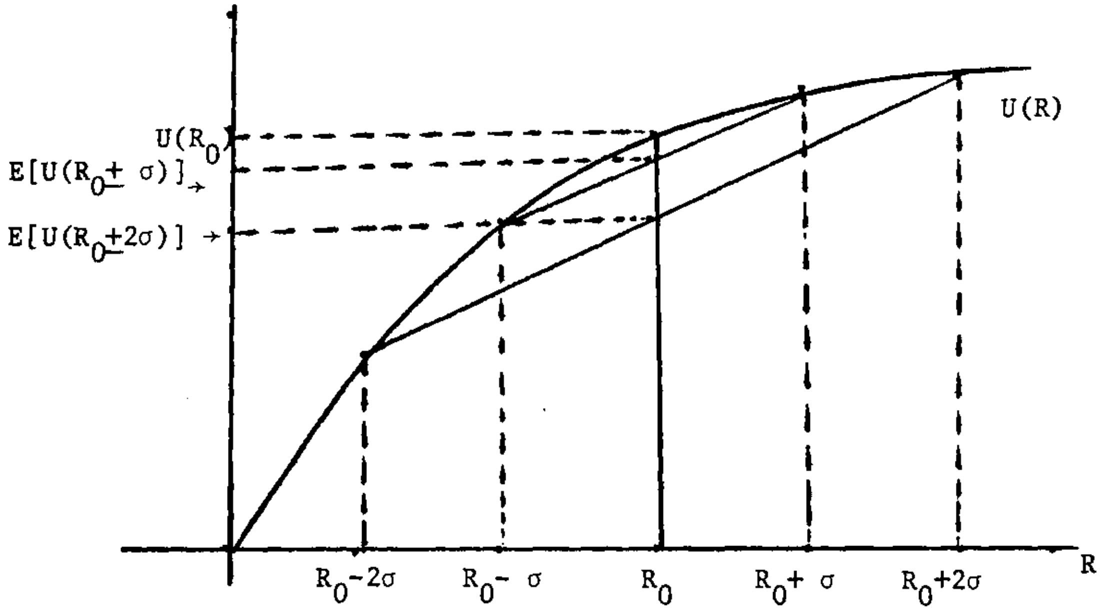


Figure 5-7. Diminishing Marginal Utility of Returns

As an example, suppose that the decision maker has two mutually exclusive projects, A1 and A2, where he can earn X_0 with certainty from Project A1, or he can invest in a risky investment which will return $X_0 + \sigma$ or $X_0 - \sigma$ with equal probability. In terms of decision tree format, this can be represented as shown in Figure 5-8.

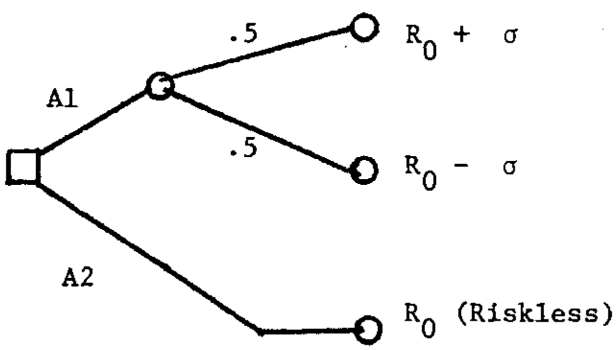


Figure 5-8. Decision Tree Example

If the expected return for Project A1 is computed, A2 has the same expected value:

$$E(R)_{A2} = (.5)(R_0 + \sigma) + (.5)(R_0 - \sigma) = R_0$$

$$VAR(R)_{A2} = \sigma^2$$

Although the expected returns are the same, a risk-averse decision maker will prefer the sure return $[E(U(R_0)) > E(U(R_0 \pm \sigma))]$. This is because the risk averter's diminishing marginal utility will cause the disutility from a return of $(R_0 - \sigma)$ to exceed the gain in utility a return of $(R_0 + \sigma)$ (i.e., $U'(R_0 - \sigma) > U'(R_0 + \sigma)$). Assume that the same decision maker receives another investment opportunity, A3, which is expected to yield either $(R_0 - 2\sigma)$ or $(R_0 + 2\sigma)$ with equal probability. The expected value and variance of this project would be

$$E(R)_{A3} = (.5)(R_0 + 2\sigma) + (.5)(R_0 - 2\sigma) = R_0$$

$$VAR(R)_{A3} = 4\sigma^2$$

In fact, A3 has greater risk (it has greater variability of return) than the other investment, A2, even though it yields the same expected return. From Figure 5.6, it is obvious that the expected utility criterion will rank the desirability of investment in the order of A1, A2 and A3 $[U(R_0) > E(U(R_0 \pm \sigma)) > E(U(R_0 \pm 2\sigma)]$.

To derive a consistent utility function which represents a risk-aversion behavior, the following two general properties must be satisfied [74]:

1. The utility function U(R) must be a monotonically increasing function of returns.
2. U(R) must possess the property of diminishing marginal utility. To determine whether marginal utility is increasing or decreasing, the slope of the utility function or the sign of the second derivative of U(R) must be examined. In other words, $U'(R) > 0$ and $U''(R) < 0$.

5.3.3.2 Logarithmic Utility Functions. To compare the expected utility criterion with the PB criterion, it is necessary to specify the utility function to be used for the expected utility criterion. However, there are numerous types of utility functions that could be selected. The utility function chosen for this study is the logarithmic utility function. The reasons that this type of utility function is utilized to compare this methodology with the project balance approach are listed below.

1. The logarithmic utility function is one of the most frequently mentioned utility functions in the literature [74, 28, 69].
2. The utility function is realistic because it exhibits positive marginal utility of returns:

$$U(A) = \log Z \qquad (Z > 0)$$

$$\frac{du(Z)}{dZ} = \frac{1}{Z} > 0$$

3. The utility function is also realistic because it has a decreasing marginal utility of returns:

$$\left[\frac{d^2U(Z)}{dZ^2} = -\frac{1}{Z^2} < 0\right]$$

This also implies that the utility function represents a decreasing risk-aversion behavior in Pratt's sense. This property has been advocated as a rational behavior for a risk-averse decision maker [74].

4. The logarithmic utility function yields an invariant preference ordering over a set of investments if the function undergoes any positive linear transformation. As an example, if one obtains a set of preference orderings from using $U(Z) = \mathrm{Log}(Z)$, then this implies that he also can obtain the same set of preference orderings by using $U(Z) = a \log(Z) + b$ ($a > 0$, b = any real number).
5. It can be transformed to create approximations of nonlogarithmic utility functions [28, p. 506]. Therefore, the conclusions obtained with the logarithmic utility function are also approximately true for some other classes of nonlinear utility functions [62, pp. 120-122].

To employ the expected utility criterion,

1. Compute utility value of the present worth associated with each cash flow realization of the probability tree for a given investment project.
2. Compute the expected utility of the project.
3. Formulate the zero-one integer programming problem.

   for $t = 0, \ldots H$

Model VI: Maximize $\sum_i (E[U])\, X_i$

subject to $\sum_i C_i X_i \leq B$

$X_i = (0,1)$

where

$E(U)_i$ = the expected utility value of project i.

It must be recognized that Model VII is an approximate formulation. The exact procedures to find the best combination of the investments which maximizes the expected utility include first deriving the probability distribution of the present value for each feasible combination, then selecting the combination which maximizes the expected value of a utility function (see Hillier [45]).

Despite its theoretical appeal, there is little use for the utility theory in capital budgeting practice [52]. The great difficulty with this approach is in specifying a utility function that can be used consistently. Furthermore, the investment decision is frequently made by a group, or committed and it is even more difficult to derive a consistent utility function which reflects group behavior [96, pp. 155].

5.3.4 Effects of Partial Information and Complete Information on the Sequential Decision Process

It must be realized that the project balance criterion will not guarantee the optimum selection of proposals that could be achieved if the decision maker had complete information about the future investment opportunities. Recall that the PB criterion is developed for those investment situations where decisions are made on a regular periodic basis. In particular, the criterion can be used for those investment situations where the decision is based on expectations about the future. Therefore, if complete information about present and future investment opportunities at the time of the current decision is available, it should be possible to select the overall optimum set of projects.

It is of interest to compare the horizon value obtained from the project balance criterion with the one obtained with complete knowledge about the future investment opportunities and the realizations of their future cash flows. Two different levels of knowledge are examined in this study. The first assumes that the decision maker has complete information about the set of investment proposals that are being considered at the time of decision. The second assumes that complete information is available regarding all the proposals being considered throughout the horizon time.

5.3.4.1 Effects of Partial Information on the Sequential Decision Process (Local Optimum with Partial Information). This case assumes that the decision maker has complete information about the set of investment proposals that are being considered at the time of decision. In other words, complete information about the investment opportunities is available only for those proposals that are currently under consideration. However, specific proposals that might originate during periods following the current period cannot be anticipated. For this investment situation, investment decisions are made sequentially as they occur. The decision criterion utilized with this perfect information is to select a set of proposals which maximizes the present worth at each decision period. Then, the local optimum for the first case is obtained from solving the following zero-one integer programming problem at each decision period.

Model VII: Maximize $\sum_i (PW)_i X_i \qquad$ for $t = 0, 1, \ldots, H$

$$\sum_i C_i X_i \leq B$$

$$X_i = (0,1)$$

where $(PW)_i$ = present worth of project i at period t.

5.3.4.2 Effects of Complete Information on the Sequential Decision Process (Global Optimum with Complete Information). This case assumes that complete information is available regarding all the proposals being considered throughout the time horizon. Since perfect information about present and future investment opportunities at the time of current decision (t = 0) is available, it should be possible to select the overall optimum set of proposals. This selection can be achieved by formulating the decision problem as the basic horizon model proposed by Weingartner [101].

Model IX: Maximize $\sum_{it} [(A_{itk})/(1+g_{it})^k]X_{it} + \ell_H$

subject to $-\sum_{it} C_{i1}X_{i1} + \ell_1 = B_0$

$$\sum_{it} C_{it}X_{it} - \ell_{t-1}(1 + i_\delta) + \ell_t = 0 \qquad t = 1, \ldots, H$$

$$X_{it} = (0,1)$$

$$\ell_t \geq 0$$

where g_{it} = growth rate (rate of return) of proposal i submitted at time t

ℓ_t = amount of money available for lending

A_{itk} = k-th cash flow occurring after horizon time for proposal i submitted at time t

As formulated in Model IX, the objective function of this horizon model maximizes the sum of all cash flows realized after the horizon discounted at their respective internal rate of return to the horizon

[90, Chap. 7]. In fact, Weingartner does not specifically suggest what kind of interest rate should be used to discount the cash flows occurring after the horizon time. He simply says that the stream of revenues associated with projects need not terminate at the time or before the horizon is discounted to the horizon at <u>some rate of interest</u> (see Weingartner [101, pp. 141]). As discussed in Section 5.2.1, the firm's objective is to maximize the accumulated capital at the horizon time; the right interest rate to be used is the respective rates of return of the proposals which do not terminate at or before the horizon (see also Section 6.5.4 for the justification).

CHAPTER VI

THE DESCRIPTION OF THE SIMULATION MODEL

In Chapter V, the PB criterion was proposed along with three methods of selecting investment alternatives. The three methods are the expected present worth criterion, the mean-variance criterion, and the expected utility criterion. These three decision criteria are to be compared with the PB criterion for the sequential decision process described in Section 5.1.

This chapter contains a detailed description of the simulation model used to test the effectiveness of these criteria. First the specific assumptions of the simulation model used in this analysis are described. Then follows the description of the simulation process which is used to test the effectiveness of those decision criteria.

Basically, the simulation model consists of two parts. The first part of the simulation model includes the generation of a periodic schedule of investment proposals (SIP) that are submitted at each decision period. In particular, detailed descriptions are given as to how the investment proposals are generated. The second part of the simulation model is the regular periodic application of the different decision criteria to the schedule of investment proposals generated in the first part. In addition, the second part of the simulation consists of the accumulation and calculation of statistics to evaluate the performance of these decision rules.

6.1 The Assumptions of the Simulation Model

In order to translate the investment decision process described in Section 5.1 into the computer simulation model, the model will require the logical flow of the decision process and assumptions that limit the model's representation of reality. The basic assumptions made in the simulation model are:

1. The firm's primary objective is to make investment decisions that promise to maximize its future worth with the limitation on funds available for investment. Although other goals also are legitimate, in order to keep the analysis manageable, these other goals are not considered in this study [96, Chap. 1]. Further, nonmonetary considerations [29] which do affect investment decisions are not considered in this study.
2. Through the normal operation of the firm, the firm makes capital investment decisions on a regular periodic basis. Each period the decision maker examines the schedule of investment proposals (SIP) submitted for consideration during that period and the investment decision is made at the end of that period. These equal successive time periods can be any time interval such as months, years, and so on.
3. The size of each proposal's first cost is assumed to be known when it is proposed, but future cash flows are random in magnitude. This assumption seems valid because for many investment proposals cash outlays are known in advance but occur either at the beginning of the proposal's life or at given times during the earlier life of the proposal. Each proposal is also assumed to have a known investment

life. The uncertain future cash flows and their probability of occurrence can be represented by a probability tree (as described in Section 4.1) in which a series of cash flows and their conditional probabilities are given. The probability tree is characterized as one which requires a net disbursement only at the beginning of its life. In other words, future cash flows in each period are not allowed to be negative (i.e., cost or outflow). The reason for this nonnegativity requirement in future cash flows is to make the current investment decision independent of the budgets at future decision times, since no precise budget situation at any future decision point in time can be predicted at the current decision time.

4. The firm bases its capital budgeting decision on the SIP for the current period and on any knowledge of prior decisions that might prove helpful. However, it is assumed that the decision maker cannot anticipate specific proposals that might originate during periods following the current period [90, Chap. 3].

5. New investments are financed exclusively through internally generated funds. In other words, the firm cannot secure external funds as part of the capital budgeting decision. However, in order to initiate the simulation, it is assumed that external funds are available only at the first few decision times of the decision process, diminishing quickly to zero as decisions are made through time. One view of these external funds is that they are receipts from investments made prior to the time the first decision is made. In accordance with this subject, Thuesen [90, Chap. 6] develops a method for determining the size of

external funds to be utilized by the simulation model for the first few periods of the simulation. This approach is used to minimize the simulation time required to approximate a steady state investment situation for a firm that is in full operation. Once the simulation progresses beyond these start-up conditions, the amount of funds available for investment in a given period t will be composed strictly of receipts received at time t from previously undertaken proposals plus the funds remaining after investments are made in the previous period. It is assumed that these remaining funds are invested at a low interest rate i_{δ}, which can be earned on highly liquid investments such as government bonds or savings accounts. These short-term investments make these funds available at any time they could be utilized.

6. No divident payments are declared throughout the operation of the firm over the study period. This policy is adopted to avoid a dampening effect on the growth of the firm and the effect that a particular dividend payment policy would have on the total capital of the firm. The purpose of this study is to examine total capital growth, and the use of these earnings is beyond the scope of this investigation.
7. Each investment proposal is considered to be an indivisible unit, and it is not possible to undertake "multiples" of any investment proposal. If a project is rejected, it is eliminated from further consideration; in other words, no projects are "postponed" in this analysis.

6.2 Description of Simulation Process

The simulation process can be separated into three phases in this study.

Phase I: The generation of a periodic schedule of investment proposals containing proposals that are submitted within a specific time interval for the decision maker's consideration at each decision period.

Phase II: The regular periodic application of the four different decision criteria described in Chapter V to the SIP generated in Phase I. In addition, the second phase of the simulation consists of the accumulation and calculation of statistics relevant to the stated purposes of this study.

Phase III: Once Phase I and II are complete, the realizations of cash flows of all the projects generated during the study period are preserved. Given these realizations of the proposal's cash flows, a global upper bound solution to the decision problem is obtained. This result is found by solving Weingartner's linear programming formulation of the horizon model [101].

Due to the degree of similarity between the investment decision process assumed in this study and the one used by Thuesen [90], some of the simulation techniques in Phase I are patterned after these tested techniques.

6.3 Generation of a Schedule of Investment Proposals

A schedule of investment proposals (SIP) is a set of investment opportunities submitted for consideration during a decision period. Due to the variability among proposals with respect to the size of investment, expected pattern of cash flows, life, and expected rate of return, it is highly unlikely that the SIP for one period would be the same as the SIP for another period. Therefore, it is reasonable to visualize a decision maker having schedules consisting of investment proposals drawn from an underlying distribution in which the size of investment, expected pattern of cash flows, life, and expected rate of return are all random variables. The existence of such a probabilistic distribution is recognized, and it is assumed to be represented by the distribution of investment opportunities with rate of return, as will be seen in Section 6.3.1.

Thus, in Phase I, to characterize each investment proposal, the simulation begins by generating the expected prospective growth rate from this distribution of investment opportunities (e.g., see Figure 6-1). Then, the type of proposal is determined by specifying the life, the size of initial investment, and the expected cash-flow pattern. These parameters are controlled by assumed probability distributions.

Once the basic characteristics of an investment proposal are defined, a probability tree showing a series of cash flows with conditional probabilities is generated. The general sequence of the simulation process involved in Phase I is shown in Figure 6.1 and details of the simulation process will be discussed according to this sequence.

6.3.1 The Distribution of Investment Opportunities with Rate of Return g_k

As defined in Section 6.3, the distribution of investment opportunities with growth rate g_k can be viewed as one which describes the average fraction f_k of dollars worth of proposals with growth g_k. This growth rate represents the internal rate of return of the proposal. That is, it is the rate which sets the receipts equal to the disbursements of the proposal.

A different schedule of investment proposals is considered each

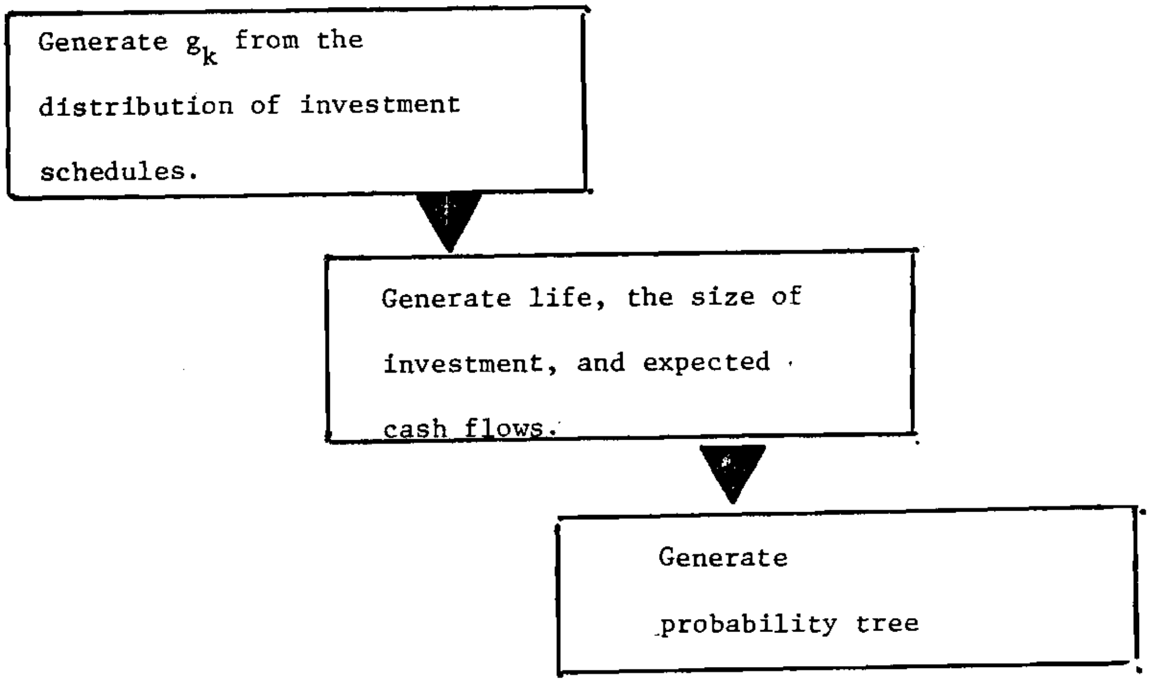


Figure 6-1. Overview of Phase I Simulation Process

period, but it is assumed that all SIP are based on this underlying distribution. Graphically, the relationship between F_k (cumulative f_k) and g_k can be displayed as in Figure 6.2 if the decision maker has the investment opportunities summarized in Table 6.1.

Another way to define the distribution of investment opportunities would be a plot of the total investment amount available at or above prospective growth g_k. However, the former approach is adopted because of its flexibility of implementation in the simulation model.

6.3.1.1 Assumptions Concerning the Changes over Time of the Distribution of Investment Opportunities with Growth Rate g_k. Since the utilization of a distribution of investment opportunities with growth rate

Table 6-1. Distribution of Investment Opportunities with Growth Rate g_k

Growth Class k	Growth Rate g_k	Fraction of Investment Opportunities in Growth Class k (f_k)	Cumulative f_k F_k
1	.30	.04	.04
2	.25	.08	.12
3	.20	.16	.28
4	.10	.28	.56
5	.06	.44	1.00

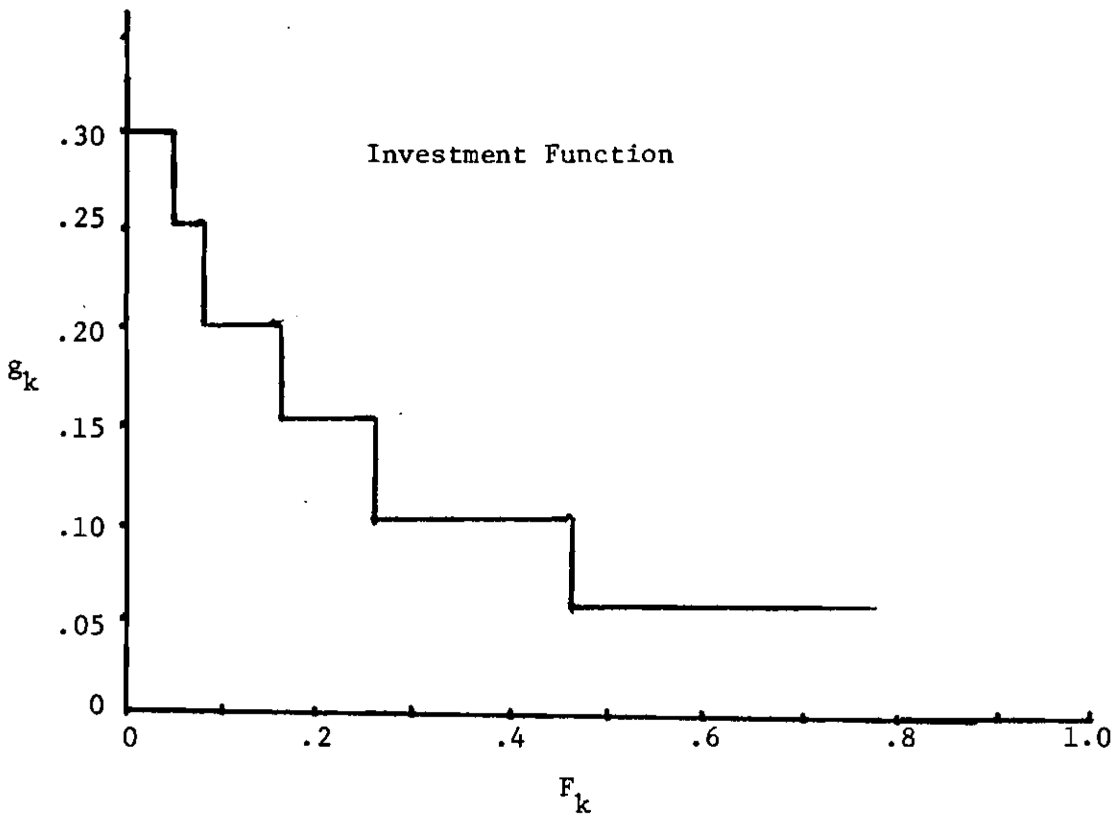


Figure 6-2. The Distribution of Investment Opportunities

g_k is basic to the method of generating proposals in the simulation, two assumptions regarding the time variations of the distribution are examined. These assumptions are used to examine the sensitivity of each decision criterion to be tested to changes in the rate of growth of the set of investment opportunities. These two investment situations are described below.

1. The distribution of investment opportunities with growth rate g_k remains constant through time. In other words, the growth rate in the total amount of investment that is available at any rate of return is constant over time [22]. This type of investment opportunity is described by Solomon [87] and others [20] and adopted in the simulation model used by Thuesen [90, Chap. 6].
2. The distribution of investment opportunities with growth rate g_k keeps growing through time. This represents the situation where the increase in investment opportunities is independent of the size of the firm. The empirical evidence of Hymer and Pashigian [47] would be supportive of such an assumption. The implications of this changing distribution of investment for the optimum investment behavior of the firm is examined by Elton [22]. When this distribution is used in the simulation model, it is assumed that the firm is exposed to more productive investment opportunities as time progresses. For either distribution of investment opportunities, the important assumption is that management's initial investment decision <u>does not</u> affect future investment opportunities [22]. Some authors [33] who have considered different assumptions about the investment schedule (that management's initial investment decision <u>affects</u> future investment opportunities) propose

a standard growth model. However, to restrict the scope of this study, the types of distribution of investment opportunities under the assumption of the standard growth model are not investigated in this study.

6.3.1.2 The Shape of the Future Schedule of Investment Opportunities. To determine whether the decision criteria are sensitive to the shape of the distribution of investment opportunities, two different types of the future schedule of investment opportunities are examined.

1. Exponential Shape: Figure 6-3 shows an exponential shape where the maximum value for any g_k (internal rate of return) is 32% with the lower limit of 6%. The shape of this distribution reflects the fact that the firm has a greater proportion of low-return proposals available than it has of high-return proposals.
2. Linear Shape: Figure 6-4 displays a linear shape in which the growth rates range from a maximum of 36% down to 6%. For this type of distribution, the proportion of proposals that are expected to be submitted with a particular prospective growth rate (internal rate of return) equals the proportion of proposals expected to be submitted for any other growth rate [90, Chap. 6]. When this type of distribution is compared to the exponential shape of distribution, the firm represented by the linear shape of distribution of investment opportunities has an increase in the proportion of good investments available for investment at high growth rates. The relationship between g_k and F_k in Figure 6-4 is expressed as

$$g_k = .36 - .30\ F_k \tag{6-1}$$

For practical reasons, the curve of the distribution of future

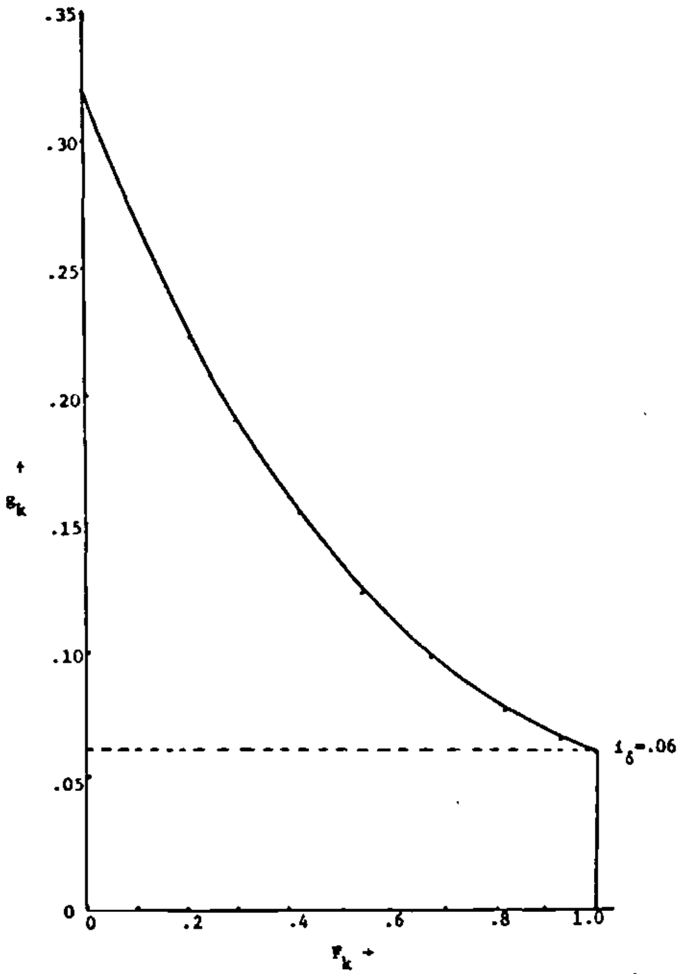


Figure 6-3. The Distribution of Investment Opportunities (Exponential Shape)

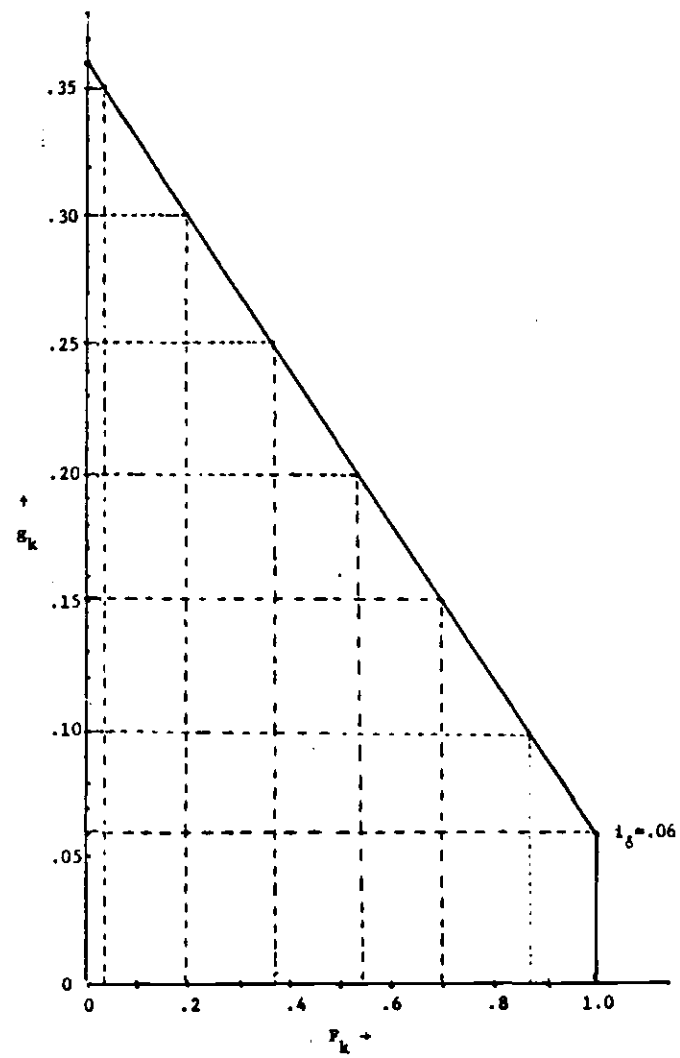


Figure 6-4. The Distribution of Investment Opportunities (Linear Shape)

investment opportunities terminates at $i_\delta = 6\%$ for either shape of distribution. In other words, the firm can invest an unlimited amount of funds at a low rate in investments that can be easily converted to cash.

Both types of the distribution of investment opportunities have been discussed in the literature. In particular, Dean [20] and Thuesen [90, Chap. 6] have recognized this relationship between investment's growth rate and the proportion of proposals submitted for consideration in terms of an exponential type of distribution. On the other hand, Thuesen [90, Chap. 6] and Elton [22] have examined a linear type of distribution of investment opportunities. In fact, the shape of the distribution of investment opportunities largely depends on the firm's expectation of future investment opportunities. Therefore, the particular relationships between g_k and F_k displayed in Figure 6-3 and Figure 6-4 were determined subjectively. However, it is the shape of these distributions that is significant, and the absolute values selected should not affect the conclusions of this study.

6.3.1.3 The Shape of the Distribution of Growing Future Investment Opportunities. As discussed in 6.3.1.1, it is of interest to examine the effects of the dynamics of the investment environment on the effectiveness of each decision criterion. The distribution of growing future investment opportunities shows an increasing proportion of good projects available for investment at a higher growth rate (internal rate of return) as time passes.

For mathematical simplicity, only the linear shape for the growing investment opportunities is considered in this study. Suppose the firm starts with the linear shape of the distribution at time $t = 0$ as shown in Figure 6-4. If θ represents the periodic growth in the proportion of growth rate g_k, then g_k for a given F_k as a function of time t would be expressed as

$$g_k(t/F_k) = .36 - (.30 - \theta t)\ F_k \tag{6-2}$$

where $$\frac{\Delta g_k(t/F_k)}{\Delta t} = \theta$$

and at $t = 0$, $g_k\ (F_k) = .36 - .30\ F_k$

The arrow in Figure 6-5 indicates the time path of the schedule of investment opportunities over the study period of 20 years. In Figure 6-5, 70% of total investments could be invested at or above g_k = 15% at t = 0. With θ = .01, however, the same 70% of total investments available at t = 10 would be invested at or above g_k = 22%. This 7% (22% - 15%) difference in the proportion of g_k represents the net increased proportion of good investments available for investment at a growth rate above g_k = 15% during a 10-year period.

In accordance with this growing pattern of the schedule of investment opportunities, it is conceivable that another possibility would be for the number of investment opportunities to be reduced over time. This pattern would represent the situation where the firm has access to special opportunities yielding more than the cost of funds, but competitive forces erode these opportunities over time [22]. However, this type of decreasing schedule of investment opportunities is of less interest; therefore, this type of schedule of investment opportunities is not examined in this study.

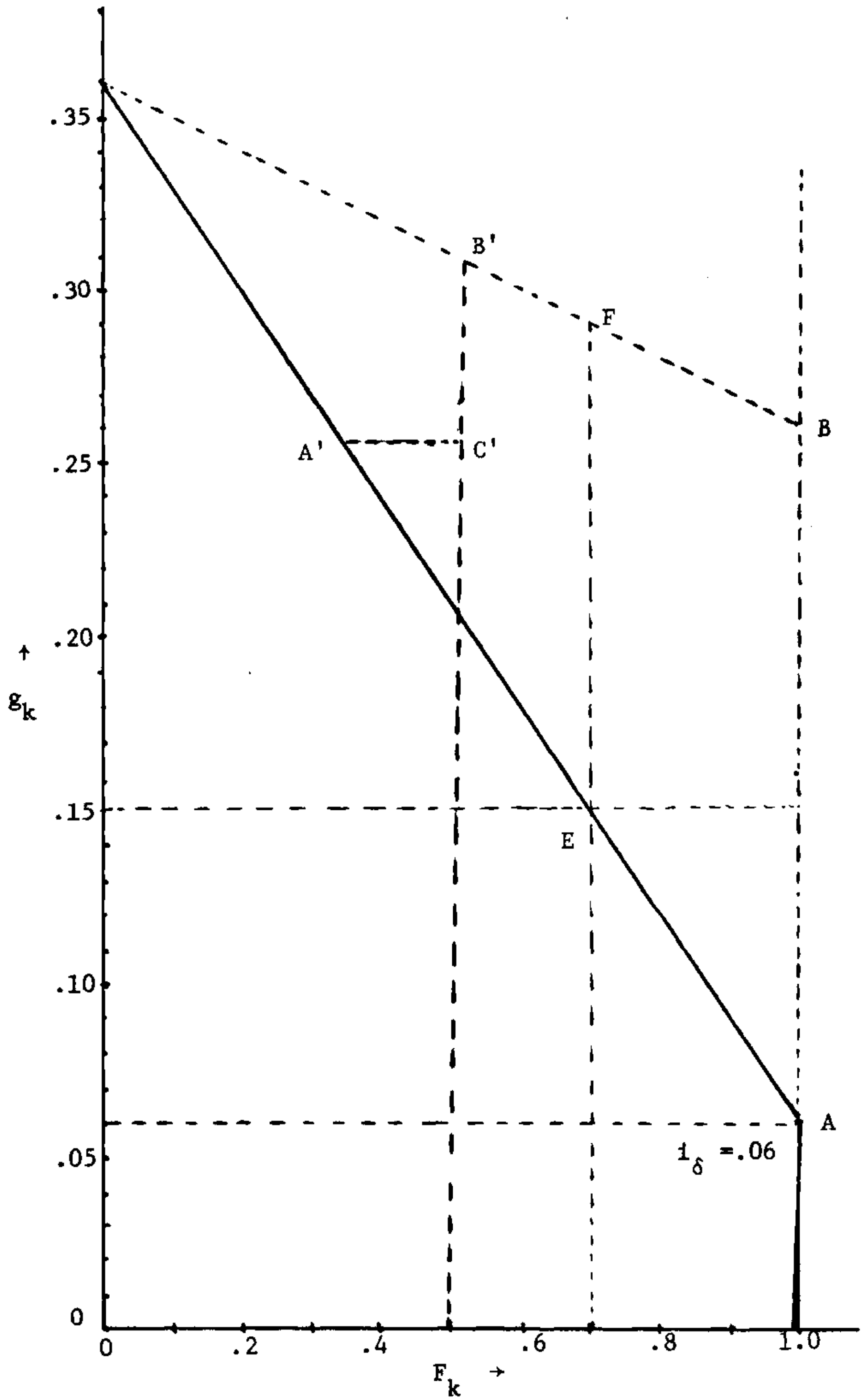


Figure 6-5. The Distribution of Growing Future Investment Opportunities

6.3.1.4 Computation of Average Capital Growth Rate ($\bar{g}$) and the Use of $\bar{g}$ in the Simulation Process. Another assumption adopted in conjunction with Thuesen's work [90] with respect to the distribution of future investment opportunities is that the total dollar value of investment opportunities grows from period to period at a rate equivalent to the average capital growth rate ($\bar{g}$). To illustrate the computation of $\bar{g}$, consider that a firm expects to allocate the funds available at decision period t to a particular set of proposals contained in the schedule of investment proposals submitted for that period. Assume a firm's expectation about the type of proposals that would be submitted for consideration for investment can be described by the distribution of investment opportunities such as shown in Figures 6-3, 6-4, and 6-5. Further assume that the firm does not invest in a proposal whose growth rate is less than the firm's minimum attractive rate of return (MARR). Then, it is possible to estimate the average capital growth rate ($\bar{g}$) of this firm by constructing an investment function which describes the expectation of how a firm's funds are to be invested.

For the exponential shape of the distribution of investment opportunities shown in Figure 6-3, an investment function can be derived for a MARR of 10%, as illustrated in Table 6-1. From Table 6-1, the average growth rate ($\bar{g}$) is computed by

$$\bar{g} = \sum_{k} (f_k)(g_k). \tag{6-3}$$

This particular investment function indicates that on the average the firm expects to have the total dollar value of investment opportunities grow from period to period at a rate equivalent to $\bar{g} = .19$.

For the linear shape of the distribution of investment opportunities shown in Figure 6-4, the computation of $\bar{g}$ is more straightforward. Since g_k is expressed as a function of F_k,

$$g_k = .36 - .30\, F_k$$

$$\bar{g} = \left[\int_{F_{g_{MARR}}}^{F_{g_{MAX}}} (.36 - .30F_k)dF_k\right] / \left[\frac{.36-g_{MARR}}{.30}\right] \tag{6-4}$$

Suppose $\quad MARR = g_{MARR} = .15$; then

$$\bar{g} = \left[\int_0^{.70} (.36 - .30F_k)\; dF_k\right]/(.70) = .255$$

Finally, the average growth rate computation for growing investment opportunities as shown in Figure 6-5 is as follows:

From Equation 6-2 and $g_k = MARR = \bar{m}$

$$F(t) = \frac{.36-g_k}{.30-\theta t} = \left[\frac{.36-\bar{m}}{.30-\theta t}\right]$$

and define $\quad K = \left[\frac{.36-\bar{m}}{\theta}\right] - .30$

then, the periodic average growth rate

$$\bar{g}(t) = \left[\int_0^{\bar{F}} (.36 - (.30 - \theta t)F)\; dF\right]\left(\frac{1}{F}\right) \tag{6-5}$$

where $$\bar{F} = \begin{cases} \left[\frac{.36-\bar{m}}{.30-\theta t}\right], & \text{if} \quad 0 \leq t \leq K \\ 1, & \text{if} \quad K < t \leq H \end{cases}$$

Thus, for Figure 6-4, if $\bar{m} = .15$ and $\theta = .01$, $K = 9$, then

$$\bar{g}(t) = \begin{cases} .255 & , \text{if } 0 \leq t \leq 9 \\ .36 - (.30 - .01t)^{1/2} & , \text{if } 9 < t \leq H \end{cases}$$

This time path of $\bar{g}(t)$ is represented by A'C'B' in Figure 6-5.

6.3.2 Types of Proposals

In this section, the general framework for Phase I simulation is described. In order to generate a particular proposal, the following five basic characteristics are defined:

(1) The interrelationships among proposals

(2) The initial investment required by the proposal

(3) The proposal life

(4) The rate of return

(5) The cash flow patterns (timing and magnitude)

6.3.2.1 Assumptions of Independence among Proposals. The proposals generated in this analysis are all considered to be functionally independent as outlined in Section 5.2.4.2. The adoption of the assumption that the proposals are independent eliminates the necessity of generating the covariances among the proposals. As discussed in Section 5.2.4.2, a meaningful assessment of covariances among the investment proposals is rather difficult to obtain for those proposals which arise in the future.

6.3.2.2 The Proposal's First Cost. In the simulation, the generation of the proposal's first cost is based on the approach taken by Thuesen [90, Chap. 6]. He generates the first cost of the proposal from a C_o distribution that is described by a mean first cost $\bar{C}_o$ and six other parameters which represent three different exponential distributions. Thus, the C_o distribution is essentially a combination of three exponential distributions placed so that the mean of the resulting C_o distribution is $\bar{C}_o$. Graphically, this relationship can be depicted as in Figure 6-6.

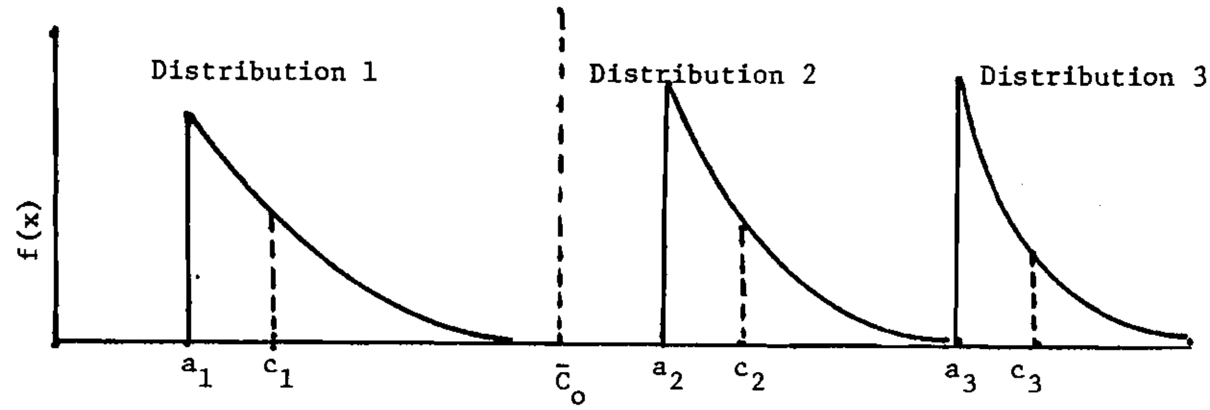


Figure 6-6. Combination of Three Exponential Distributions

In Figure 6-6, the three exponential distributions correspond to

$$f(x) = (1/(c_i - a_i))e^{-(1/(c_i - a_i))(x - a_i)}$$

where $x > a_i$, $i = 1,2,3$

and the cumulative distribution F(x)

$$F(x) = 1 - e^{-(1/(c_i - a_i))(x - a_i)}$$

Thus, x can be viewed as

$$x = a_i - (c_i - a_i)\ell n(1 - F(x)) \tag{6-6}$$

Therefore, by specifying a_i and c_i, an exponential distribution can be placed anywhere on the x axis. By placing three such distributions on one axis, and by sampling from the three distributions an appropriate freaction of the time, it is possible to have the expected value of all the sampling equal to $\bar{C}_o$. To make sure the samples drawn in this way represent those from the C_o distribution, Thuesen provides the following conditions that must be held between those distribution parameters and the fraction of time (f_i) one should sample from distribution i (i = 1,2,3):

$$\bar{C}_o = f_1C_1 + f_2C_2 + f_3c_3 \tag{6-7}$$

$$f_1 + f_2 + f_3 = 1$$

$$c_1f_1 = c_2f_2 \text{ and } c_2 \leq C_o \leq c_3$$

Two reasons are stated for using this rather complicated scheme in the generation of the proposal's first cost.

1. The approximate exponential shape that results from this combination would generate a greater proportion of proposals with smaller first costs. This property is desirable because the number of smaller proposals is usually greater than the number of large proposals in most capital budgeting situations.

2. This approach makes it possible to extend the range of C_o in the sample with relative ease. Thus, it is possible to have some reasonable probability of selecting a C_o that is rather large relative to $\bar{C}_o$. As an example, the probability of observing a value C_o three times the mean in a single exponential is approximately .05, which is deemed too small for this type of simulation study.

As the total capital of the firm being simulated grows from period to period, it is logical to expect that the funds available for investment also grow from period to period. Thus, if the average cost ($\bar{C}_o$) of the investment proposals and the distribution related to the number of proposals per period are held constant, the budget after some periods of operation would be large enough to undertake all the proposals contained in the SIP (that is, if they meet the minimum conditions set by each decision criterion). If this is the case, the effect of using different criteria becomes obscured by the fact that the decision rules do not have to discriminate when there is no financial constraint. It was decided to hold the number of proposals constant and <u>to have the individual proposal's first cost grow at the expected growth rate of total capital, $\bar{g}$</u>, so that it would be possible to maintain a reasonable balance between the funds available for investment and the total first cost of investments in each period's SIP. An algorithm for generating the proposal's first cost at decision period t is shown in Figure 6-7.

<u>6.3.2.3 Proposal Life</u>. In practice, it is common to observe that there are usually more investment proposals with short lives than with long lives. Therefore, the proposal life is generated from a single exponential distribution. However, when generating a probability tree in

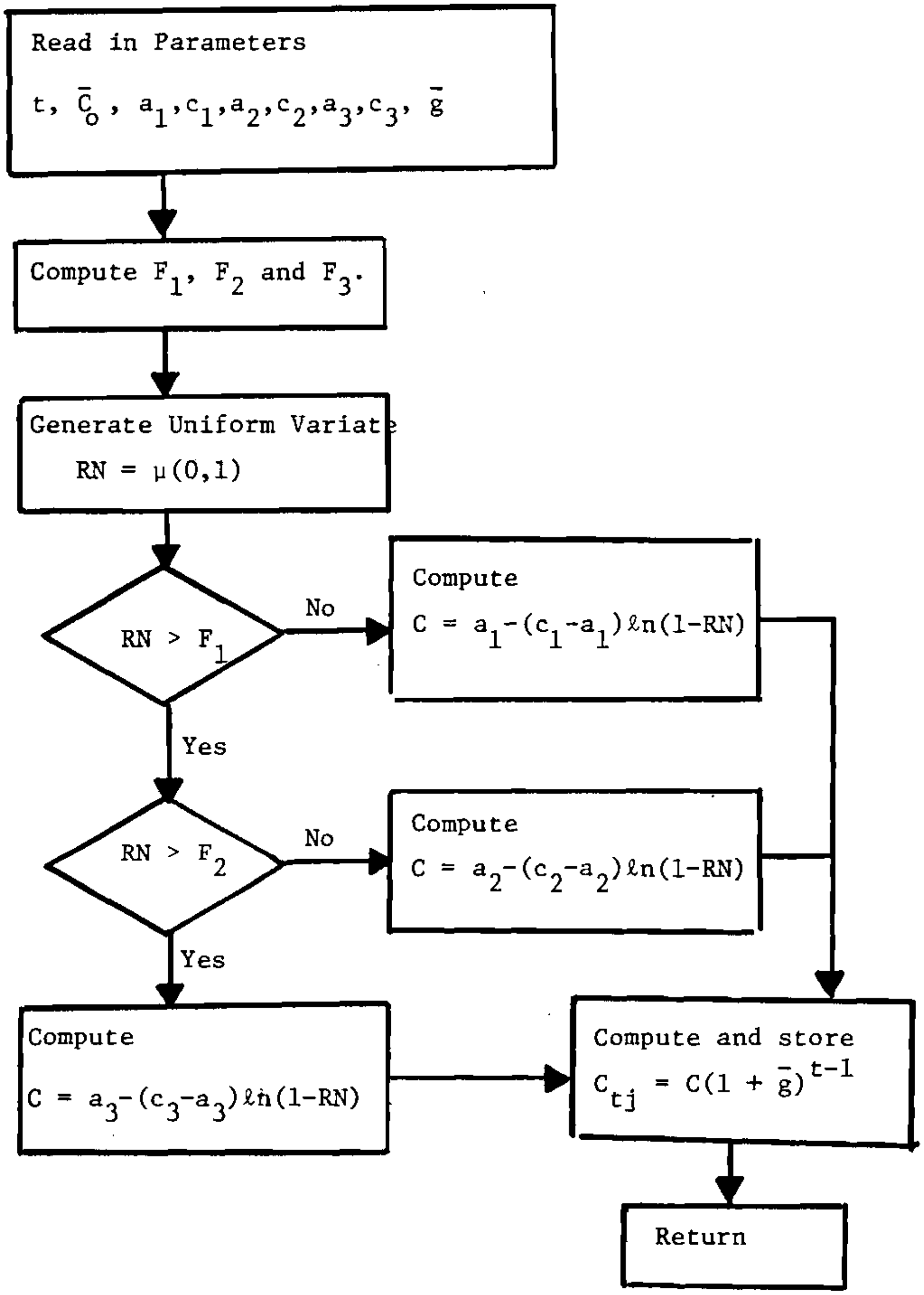


Figure 6-7. Logic to Generate The Proposal's First Cost

Section 6.3.3, the proposal life becomes a critical factor. This is because the total number of branches to be generated in a probability tree increase exponentially as the life of the proposal increases. For example, if the proposal starts with two chance elements at the outset and each node has two elements, then the total number of tree branches for a 15-year life would be 2^{15}(=32768). But the total number of elements would be $\sum_k 2^k$, where k = 1,2,...,15. The computer time and memory space required to generate and store the probability tree for a long-lived project is impractical. Since a large number of proposals are generated throughout the study period, it is necessary to limit the maximum life which a proposal can take at L_{max}. Thus, three parameters are used to define this truncated exponential distribution (L_{min}, $\bar{n}$, L_{max}).

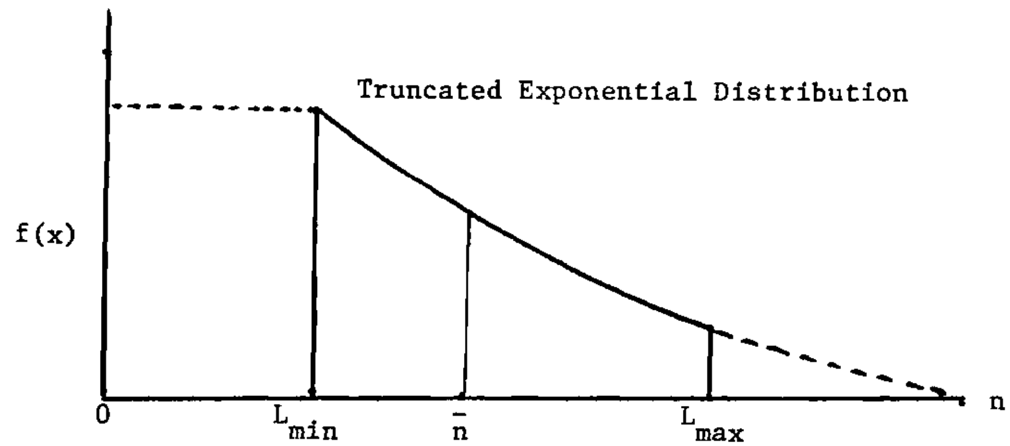


Figure 6-8. Average Life Distribution

Let L_{max} = the maximum proposal life specified in the shifted life distribution

$\bar{n}$ = the average proposal life

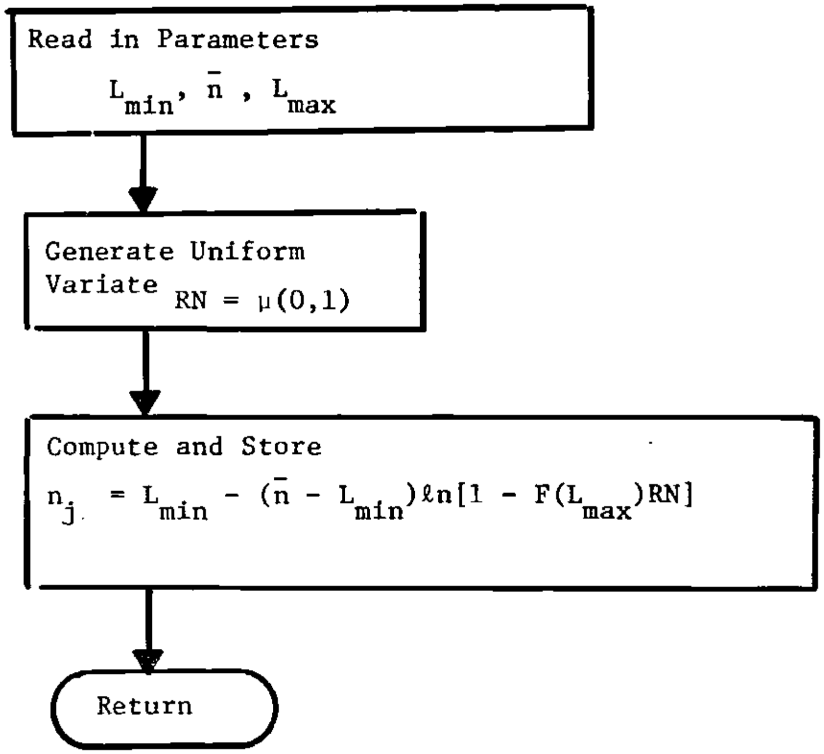


Figure 6-9. Logic to Generate the Life of Proposal j

L_{min} = the maximum proposal life allowed in the life distribution

Without the truncated life,

$$F(n) = [1 - e^{-(1/(\bar{n}-L_{min}))(n-L_{min})}]$$

Then, the truncated $\bar{F}(n)$ would be

$$F(n) = (1/F(L_{max}))[1 - e^{-(1/(\bar{n}-L_{min}))(n-L_{min})}]$$

Thus, the proposal life is determined by

$$n = L_{min} - [\bar{n} - L_{min}]\ell n[1 - F(L_{max})F(n)] \qquad (6\text{-}8)$$

This probability distribution is considered to remain constant throughout the study period. An algorithm for generating the proposal life n is shown in Figure 6-9.

6.3.2.4 Expected Rate of Return for a Proposal $E[g_k]$. In the simulation, the expected growth rate $E[g_k]$ for a proposal is selected through the following steps:

1. Identify the investment environment (constant versus growing distribution of investment opportunities).
2. Identify the type of the distribution of investment opportunities (exponential versis linear).
3. Generate a uniform random variate $\mu(0,1)$ = RN.
4. Determine g_k (see Section 6.3.1):

If the distribution of investment opportunities is exponential in shape, this selection is easily accomplished. Since the distribution of investment opportunities with g_k is shown as a cumulative distribution, the Monte Carlo sampling can be made directly. If the distribution of investment opportunities is a linear shape, then it is defined as follows:

For constant investment opportunities:

$$g_k = .36 - .30(RN)$$

For growing investment opportunities:

$$g_k = .36 - (.30 -.10(t-1)(RN))$$

The logic to generate g_k is shown in Figure 6-10.

6.3.2.5 Expected Cash-Flow Pattern. In order to generate the series of cash flows to be represented by a probability tree, it is necessary to identify the expected pattern of a proposal's cash-flow receipts series. Once a proposal's first cost C_{jo}, its life n_j, and its rate of return g_j are known, it is possible to determine the expected amount and timing of its cash flows, provided its cash-flow pattern is known, In this simulation, four basic cash-flow patterns are used to generate the probabilistic cash flows:

(1) Single Payment

(2) Uniform Series

(3) Gradient Series (Increasing)

(4) Gradient Series (Decreasing)

By using combinations of the gradient series patterns and the uniform series pattern, it is possible to generate an unlimited number of

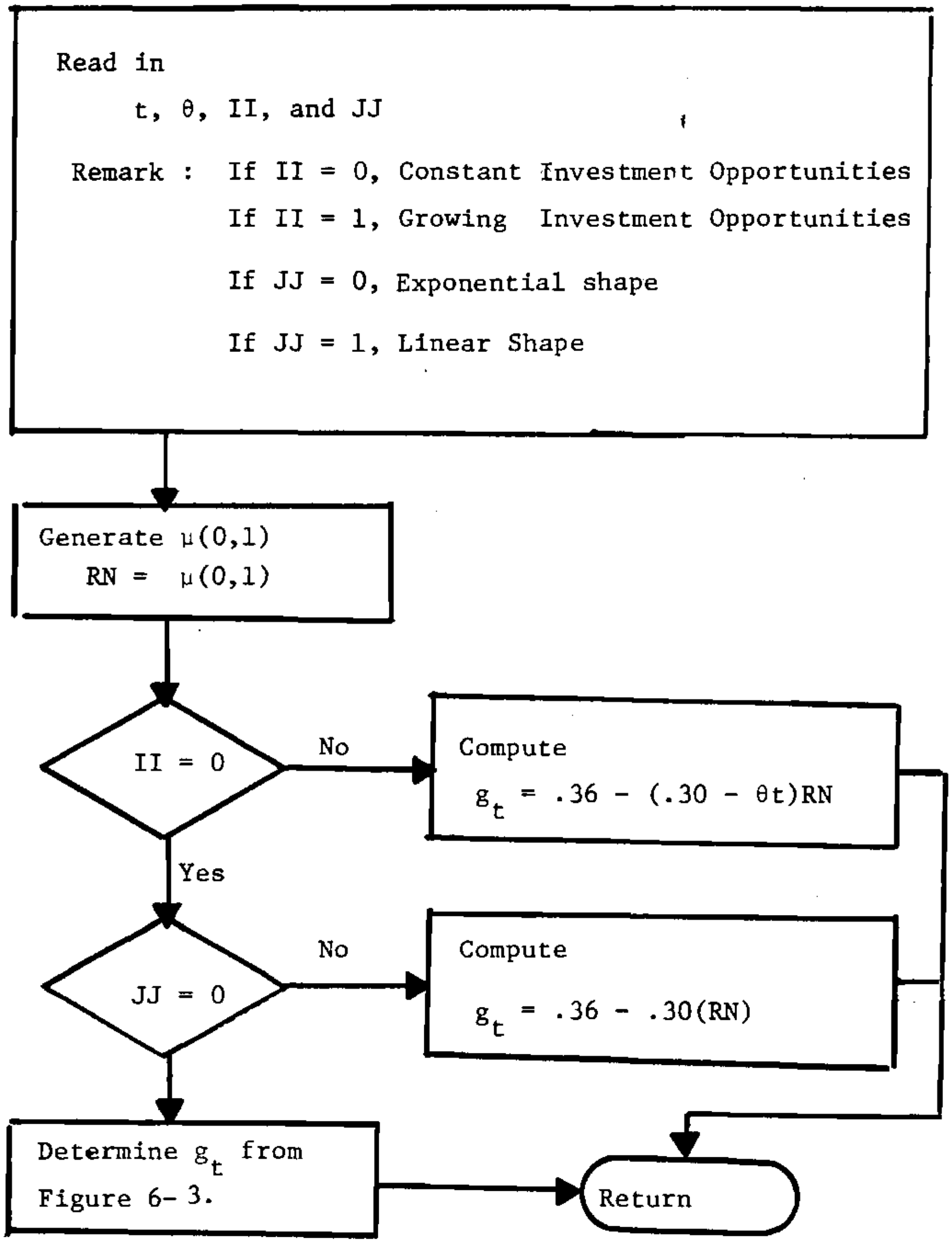


Figure 6-10. The Logic to Generate g_t

variations of these patterns in the simulation. The description of how the uniform series and gradient series are combined is given as follows:

1. Combination of the uniform series and increasing gradient series

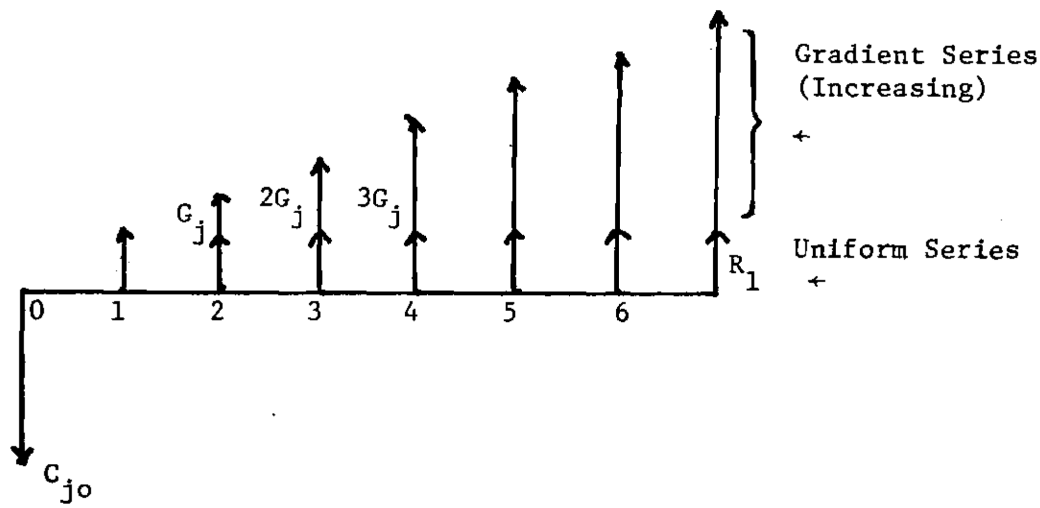


Figure 6-11. Combination of the Uniform Series and Increasing Gradient Series

2. Combination of the uniform series and decreasing gradient series

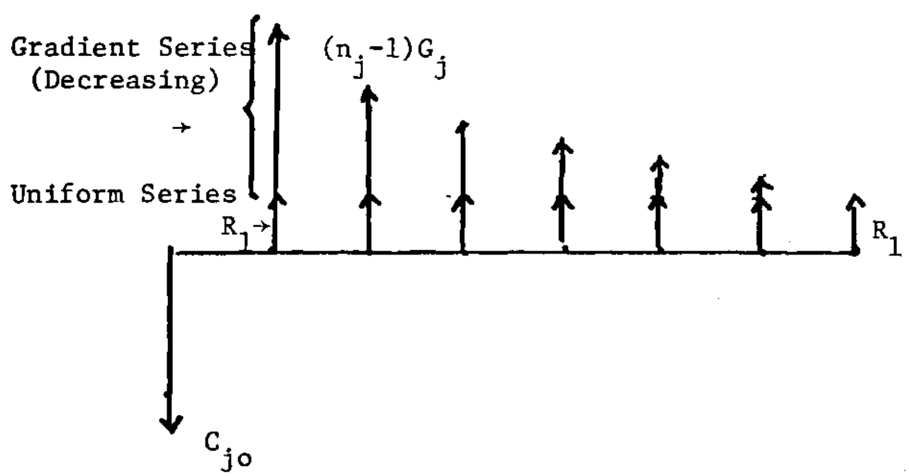


Figure 6-12. Combination (Uniform Series + Gradient Series)

It is important to recognize that the cash-flow patterns in Figure 6-11 and Figure 6-12 can be obtained from a combination of two uniform series cash flows such as shown in Figure 6-13

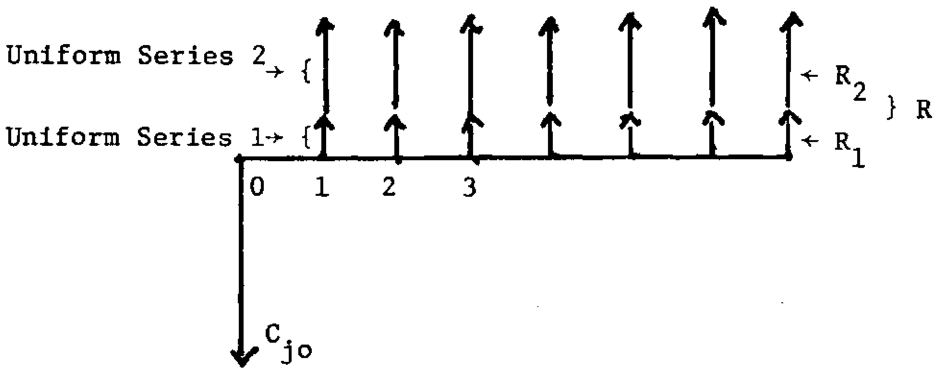


Figure 6-13. Combination Uniform Series

The conversion of the R_2 segment of the uniform series in Figure 6-13 to the increasing or decreasing portion of the series shown in Figure 6-11 or Figure 6-12 requires the project's rate of return. The computations that transform Figure 6-13 into either Figure 6-11 or Figure 6-12 are as follows:

For increasing gradient series:

$$R_2 = G_j(A/G\ g_j,\ n_j) \qquad G_j = R_2/(A/G\ g_j,\ n_j)$$

where $$(A/G\ g_j, n_j) = \left[\frac{1}{g_j} - \frac{n_j}{(1+g_j)^{n_j}-1}\right]$$

For decreasing gradient series:

$$R_2 = (n_j - 1)G_j - G_j(^{A/G}\ g_j,\ n_j),\ G_j = R_2/(n_j - 1 - (^{A/G}\ g_j, n_j)$$

Therefore, a variety of these combinations can be achieves by controlling the size of R_2 relative to R, where $R = R_1 + R_2 = C_{jo}(^{A/P}\ g_j, n_j)$ and

$$(^{A/P}\ g_j, n_j) = [\frac{g_j(1+g_j)^{n_j}}{(1+g_j)^{n_j} - 1}]$$

See Figure 6-13. It is evident that the larger R_2 is relative to R, the larger the gradient series portion of theseries shown in Figure 6-11 and Figure 6-12. The smaller R_2, the closer the series of Figure 6-11 and Figure 6-12 will approach to a uniform series.

In the simulation a particular cash-flow pattern is randomly generated for each proposal from a predetermined distribution of cash-flow shapes. If the cash-flow pattern selected is a gradient series cash flow, a random choice is made between the increasing series and the decreasing series. Then the value of R_2 relative to R is determined in the simulation by a fraction f_R that is a random variable such that

$$0 \leq f_R \leq 1$$

and that

$$R_1 = f_R R$$

$$R_2 = (1 - f_R)R$$

Thus, when f_R is selected for either of the two combination series, the following distinctive cash-flow patterns result for the particular values of f_R shown below:

If $f_R = 0.$, the resulting series is strictly a gradient series

If $f_R = 1.$, the resulting series is strictly a uniform series

If $0 < f_R < 1.$, the resulting series is a combination of a uniform series and a gradient series

Symbolically,

Q_1 = the probability of observing a single payment from the cash flow distribution

$Q_1' = 1 - Q_1$ = probability of a series payment type cash flow

Q_2 = the probability of the cash flow being a combination of decreasing series, if the proposal is a series payment type cash flow

$Q_2' = 1 - Q_2$ = the probability of the cash flow being a combination of increasing series

The computations of the expected cash-flow series for a single proposal j are given in Equation 6-9 and the logic to generate $E[F_t]_j$ is shown in Figure 6-14.

$$E[F_t]_j = R_{1j} + F_j(t-1), \text{ for increasing series}$$

$$E[F_t]_j = R_{1j} + (n_j - 1)G_j - G_j(t-1), \text{ for decreasing series} \tag{6-9}$$

6.3.3 The Generation of Probabilistic Cash Flows

This section illustrates the procedures for constructing a probability tree for a given proposal based on the five basic characteristics defined in Section 6.3.2.

6.3.3.1 The Number of Elements Originating from Each Node in a Probability Tree. To illustrate what is meant by the number of elements

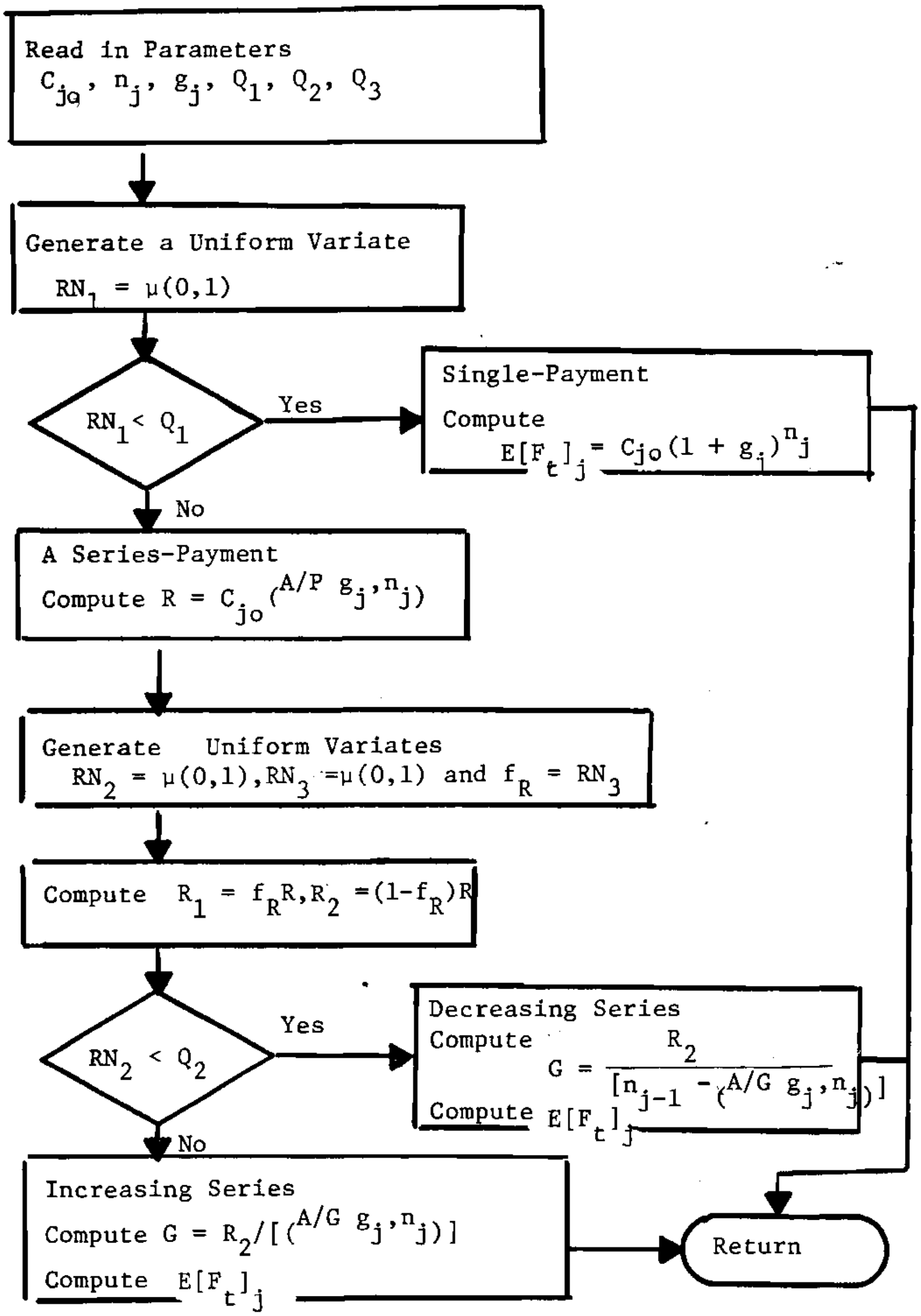


Figure 6-14. Logic to Generate Expected Cash Flow Series for Proposal j

originating from a node, consider a proposal having a two-year life as shown in Figure 6-15. The proposal would have two possible receipts, F_{11} and F_{12}, at the end of Period 1. These two possible paths of realization of cash receipts are defined as two elements originating from Node 01. If at the end of Period 1, element F_{11} is realized, then Node 11 in Figure 6-15 is assumed to anticipate two possible elements at the end of Period 2 (F_{21} and F_{22}). In the same manner, if F_{12} is realized at the end of Period 1, Node 12 is assumed to have only two probable cash receipts, F_{23} and F_{24}, at the end of Period 2. Therefore, the total number of elements in the probability tree equal 6 ($= \sum_{i=1}^{n} 2^i$) and the total number of branches which is a sequence of realization of elements, becomes 4.

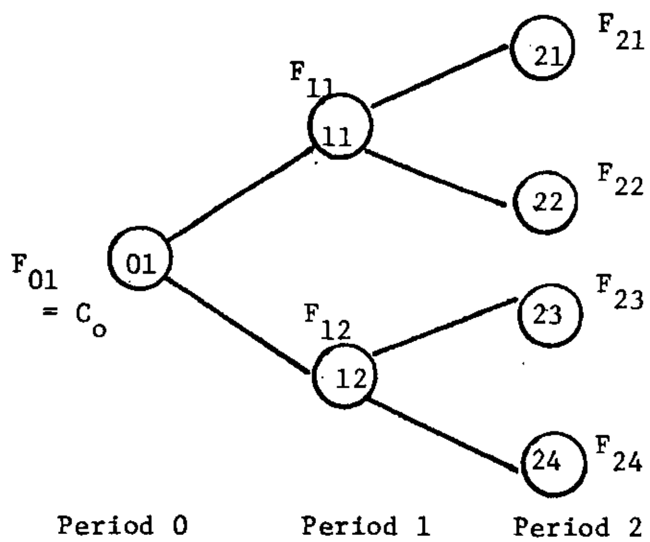


Figure 6-15. Probability Tree with Two-Year Investment Life

As shown above, it becomes evident that the total number of elements in a probability tree with an investment life of n years increases at the rate of $Y = \sum_{i=1}^{n} \gamma^{i}$, where γ is defined as the number of elements originating from a single chance node. For example, a proposal with a 10-year life will have to generate 1,024 branches with elements of $\sum_{i=1}^{10} 2^{i}$ in its probability tree. If every node in the probability tree for the proposal has three elements rather than two, then the total number of elements to be generated would be $\sum_{i=1}^{10} 3^{i}$. Thus, it is evident that the larger value of "γ," the greater the complexity of the tree with the computation quickly becoming impractical.

The decision of how many elements emanating from each node should be made is closely related to the determination of variability of the probability distribution of return at that chance node. In a probability tree, the addition of a large number of elements can serve to represent any empirical probability distribution which approximates a chance event occurring at that particular node. For example, in Figure 6-16, chance event Node 12 can be made more to closely approximate the desired continuous probability distribution by increasing the number of elements as shown in Figure 6-16(a) [41]. As the number of elements decrease, a greater variability in the empirical probability distribution is expected. Thus, when a probability tree is constructed based on a smaller value of "γ," the value of knowing the realization of a particular sequence of cash flow will be more pronounced. Therefore, in this study, the lowest practical level for the value of "γ" at a node is set to two.

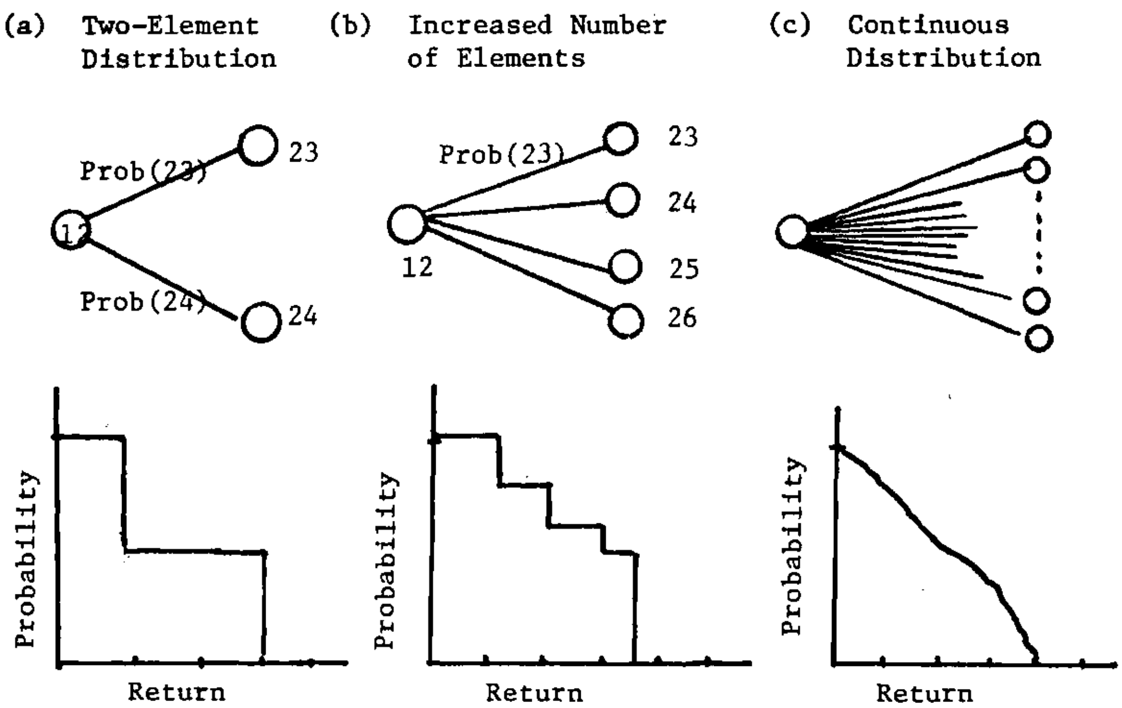


Figure 6-16. Probability Distributions at Chance Nodes

6.3.3.2 Probability Assessments at Chance Nodes. The probability assessment at each node is carried out by generating a uniform random variate (P) and assigning this probability to one element originating at that chance node and the complementary probability (1-P) to the other element. This is because at each chance node the sum of the probabilities of occurring elements must be equal to 1.

In the simulation, for the probability tree shown in Figure 6-15, a uniform random deviate U_1 is generated at Node 01; then the values U_1 and $1 - U_1$ are stored as the probabilities of realizing cash receipts F_{11} and F_{12}, respectively. Consequently, at Node 11 another uniform random deviate U_2 is generated and assigned to cash receipt F_{21} as the conditional probability that element F_{21} will be realized at the end of

Period 2. Then the value of $1 - U_2$ is assigned to element F_{22} as the conditional probability that element F_{22} will be realized at the end of Period 2. In the same manner, at Node 12, a uniform random variate U_3 is generated and stored as the conditional probability of realizing element F_{23} for given Node 12 and so on.

6.3.3.3 Assessments of Cash Flows at Chance Nodes. Once a proposal's first cost C_{jo}, its life n_j, and its conditional probabilities in the probability tree are determined, the final task is to assign the cash receipts to the chance nodes in the probability tree. In order to simulate the magnitude of the random cash receipts occurring at chance nodes in the probability tree, a knowledge of the probability distributions of the cash receipts occurring at those chance nodes is required.

More specifically, let F_t be the random variable which takes on the value of the net cash flow during the t^{th} year, where $t = 0,1,2,\ldots,n$. In the simulation, F_t is assumed to have a normal distribution with a known mean, $E[F_t]$, and known standard deviation, σ_{F_t}. A normal distribution is used because of its simplicity; it requires only two parameters to describe fully the characteristics of the distribution.

It is recognized that these assumptions regarding F_t often will not be completely justified. In fact, depending on the various investment situations, a number of different probability distributions could be used to determine the cash receipts. However, as indicated by Hillier [43], it would seem that, for many types of prospective cash flows, the best subjective probability distribution would be a nearly symmetrical one resembling the normal distribution. Further, Hillier points out the fact that by the Central Limit Theorem, the actual distribution of F_t can sometimes deviate considerably from the normal distribution without significantly affecting the final results.

In Section 6.3.2.5, the expected cash-flow series $E[F_t]$ is computed based on the proposal's first cost C_{jo}, its life, and its rate or return g_j. Therefore, by utilizing $E[F_t]$ as the mean of the distribution of cash receipts occurring at the chance nodes of period t and by specifying the appropriate variance, it is possible to generate a unique probability tree for each proposal.

1. Series Payment. To illustrate the basic simulation techniques involved in the generation of cash receipts at the chance nodes, consider a proposal whose first cost, life, expected cash flows and conditional probabilities are given in a probability tree as shown in Figure 6-17.

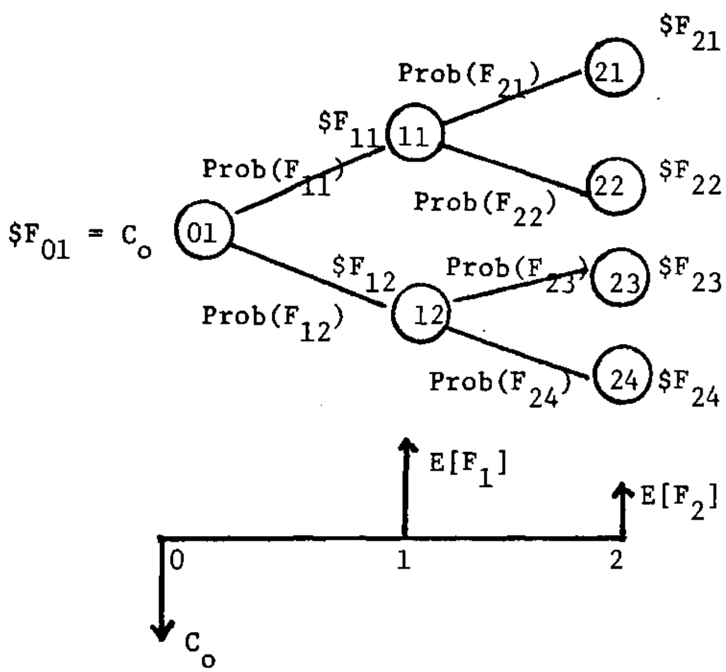


Figure 6-17. Probability Tree and Expected Cash Flows with Two-Year's Investment Life

The standard deviations of $E[F_t]$ are specified in terms of a percentage of the respective mean $E[F_t]$ as shown in Equation 6-10. It is also recognized that the assumption regarding the relationship between the mean and the standard deviation in Equation 6-10 will not be completely justified. However, it would be reasonable to assume that greater variability would often accompany higher mean values (see Levy [53]), and if this is the case, the use of Equation 6-10 would be one of the simplest ways to define the relationship between the mean and the standard deviation with respect to the size of the cash receipt.

$$\sigma_{F_t} = \beta E[F_t] \qquad t = 1,2,\ldots,n \tag{6-10}$$

Then, for t = 0, F_{01} at Node 01 represents the proposal's first cost, which is assumed to be known with certainty. Thus F_{01} is C_o itself.

$$F_{01} = C_o$$

For t = 1, there are two elements (F_{11} and F_{12}) which can be assumed to be distributed normally with $\eta(E[F_1], \sigma^2_{F_1})$. In the simulation, two normal random deviate $\eta_1(0,1)$ and $\eta_2(0,1)$ are generated and the events F_{11} and F_{12} are computed as

$$F_{11} = E[F_1] + \sigma_{F_1}(\eta_1(0,1))$$

$$F_{12} = E[F_1] + \sigma_{F_1}(\eta_2(0,1))$$

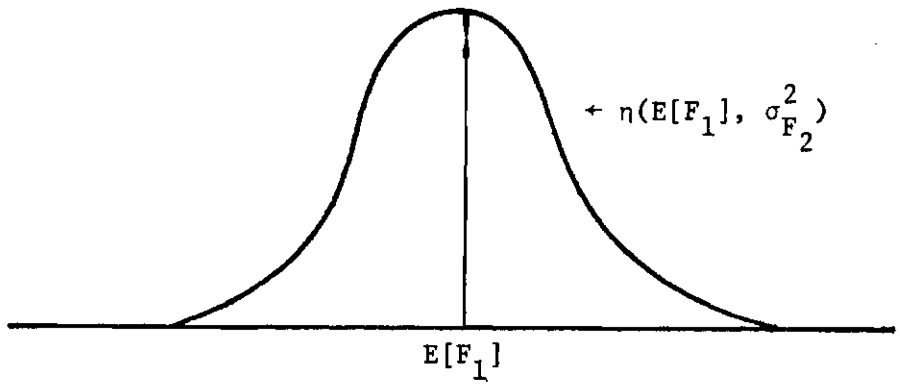


Figure 6-18. The Sampling Distribution at Period One

For t = 2, there are four elements (F_{21}, F_{22}, F_{23}, F_{24}) which are also assumed to be normally distributed with $\eta(E[F_2], \sigma^2_{F_2})$. Therefore, four samples are drawn from the distribution of $\eta(E[F_2], \sigma^2_{F_2})$; they are:

$$F_{21} = E[F_2] + \sigma_{F_2} (\eta_3(0,1))$$

$$F_{22} = E[F_2] + \sigma_{F_2} (\eta_4(0,1))$$

$$F_{23} = E[F_2] + \sigma_{F_2} (\eta_5(0,1))$$

$$F_{24} = E[F_2] + \sigma_{F_2} (\eta_6(0,1))$$

The validity of using this approach to generate the random cash receipts at the chance nodes in the probability tree can be easily supported. For example in Figure 6-17, suppose that Branch 1 (01 → 11 → 21) is realized. Then, this path is equivalent to realizing the sequence of cash flows $F_{01} \to F_{11} \to F_{21}$. Before assigning any values to this sequence of cash flows, F_{11} and F_{21} are random variables. However F_{11} and F_{21} are to be drawn from the distributions of $\eta(E[F_1], \sigma^2_{F_1})$ and $\eta(E[F_2], \sigma^2_{F_2})$, respectively. Then, the expected cash-flow series for this branch would be $F_{01} = C_o$ $E[F_{11}] \to E[F_{21}]$, which in turn is equivalent to $C_o \to E[F_1] \to E[F_2]$. The same argument applies to Branch 2, Branch 3, and Branch 4 such that the probability tree generated from this simulation technique is expected to have a sequence of cash flow of $C_o \to E[F_1] \to E[F_2]$. This approach proved to be very successful in the generation of probability trees for the simulation model.

2. <u>Single Payment</u>: The generation of a probability tree for a proposal with a single payment is a special case of series payment. Since the cash receipt occurs only at the end of life, the shape of the associated probability tree is depicted in Figure 6-19.

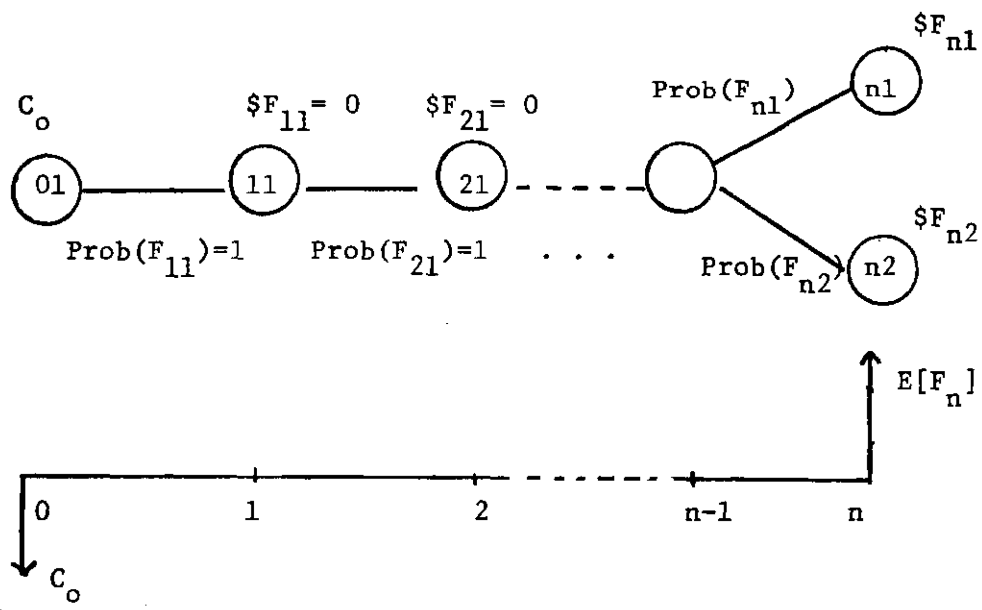


Figure 6-19. Probability Tree Associated with Single Payment

The basic procedure involved in the generation of the type of probability tree shown in Figure 6-19 can be outlined as follows:

a) Probability Assessment

If $t \leq n - 1$, probability $(F_{t1}) = 1$

If $t = n$, generate a uniform random deviate $\mu(0,1)$ and let

$$\text{Prob}(F_{n1}) = \mu(0,1)$$

$$\text{Prob}(F_{n2}) = 1 - \mu(0,1)$$

b) Cash Flow Assessment

If $t = 0$, $F_{01} = C_o$

If $0 < t \leq n-2$, $F_{t1} = 0$

If $t = n-1$, generate two normal random deviates $\eta_1(0,1)$ and $\eta_2(0,1)$, and let

$$F_{n1} = E[F_n] + \sigma_{F_n}(\eta_1(0,1))$$

$$F_{n2} = E[F_n] + \sigma_{F_n}(\eta_2(0,1))$$

An algorithm to generate a probability tree is shown in Figure 6-20, and the detailed simulation logic for Phase I simulation is also given in Figure 6-21.

6.3.4 The Number of Proposals per Period

The number of proposals per decision period for a particular firm may be a constant or a random variable. If the number of proposals is considered to be a random variable, a normal distribution truncated at the left may be used because the minimum number of proposals to be considered in a given period is considered to be one.

Because the time to simulate the probability trees and to solve a

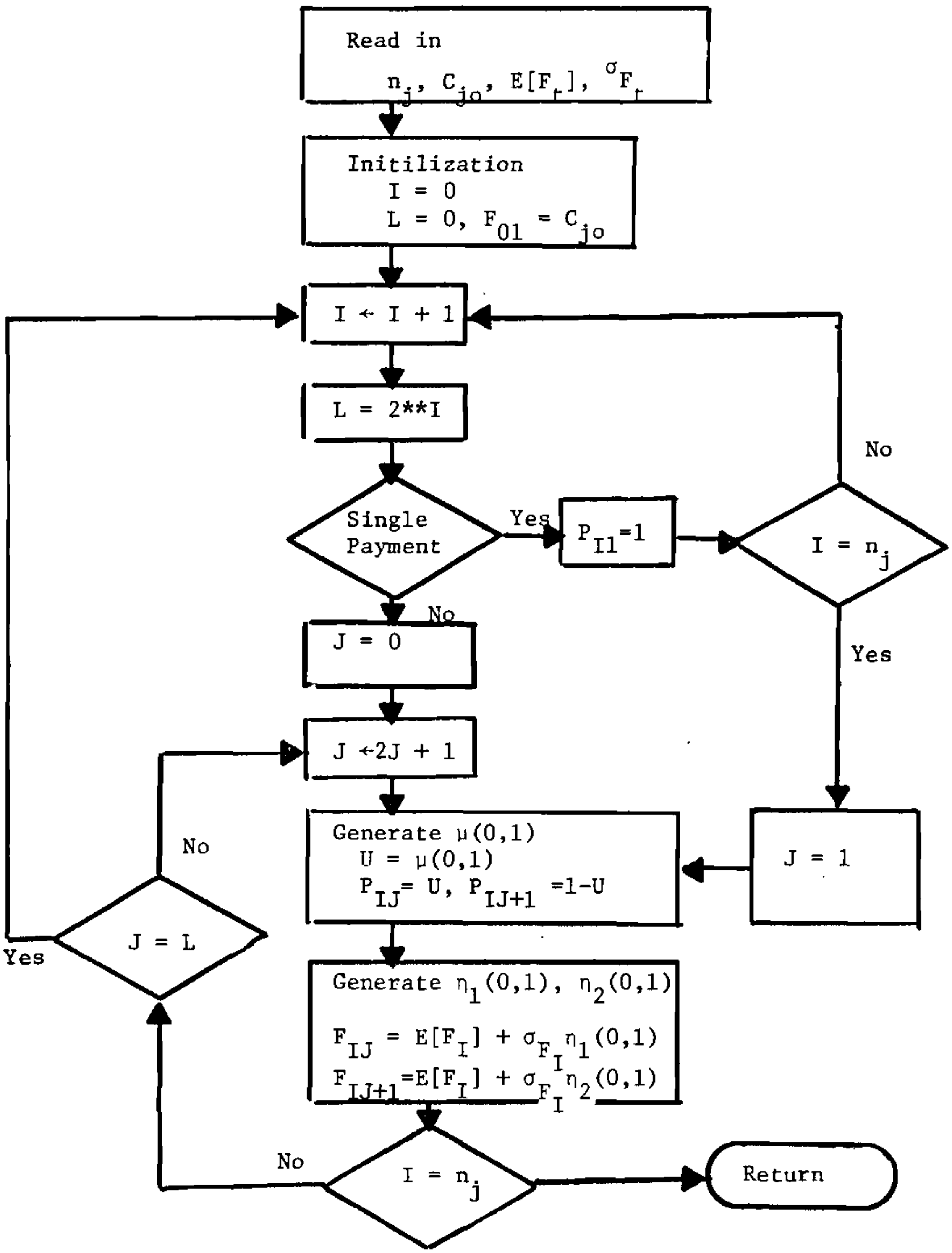


Figure 6-20. Logic to Generate a Probability Tree

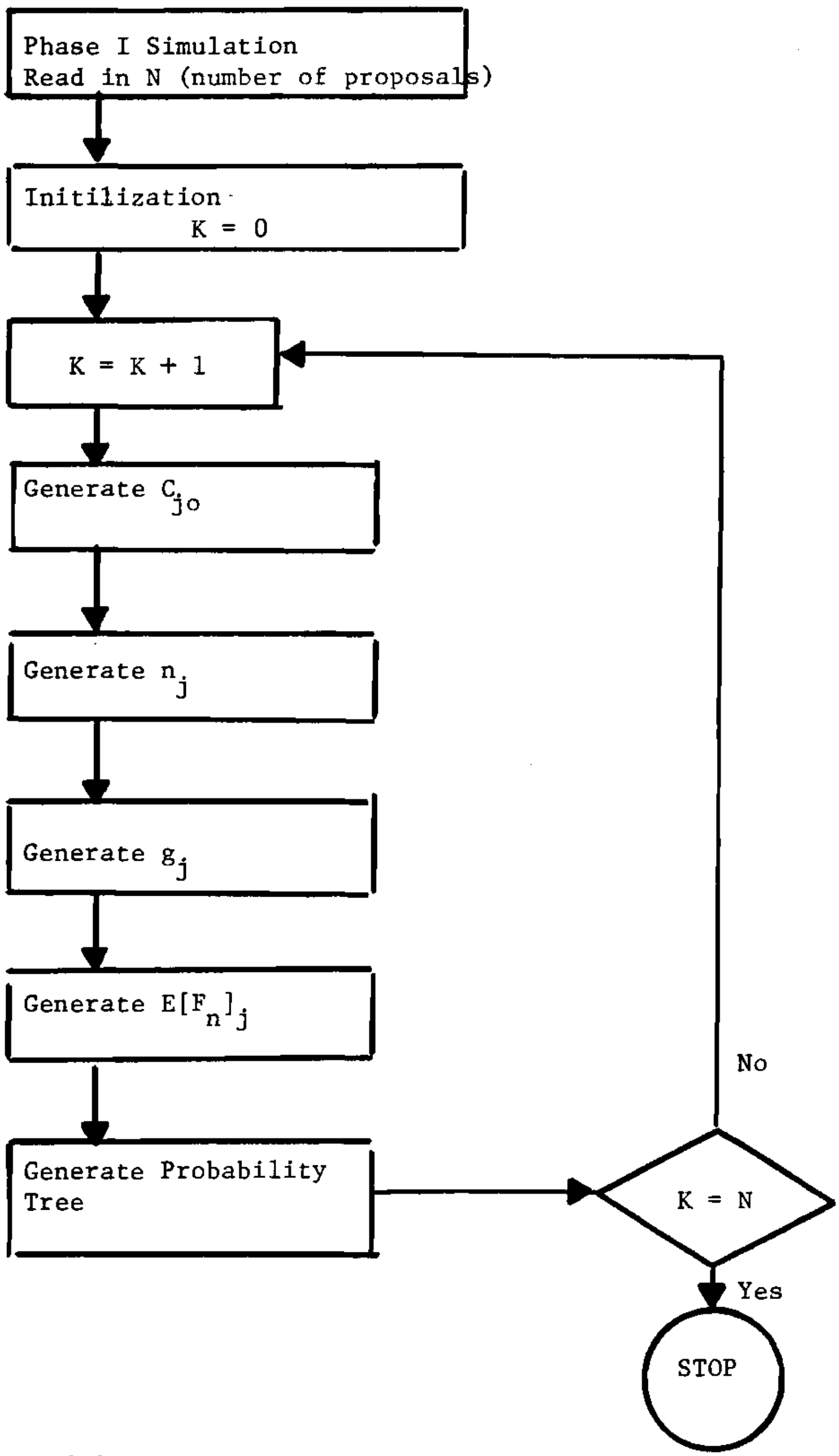


Figure 6-21. The Detailed Simulation Logic for Phase I Simulation

linear integer programming code increases significantly with the number of proposals in a SIP, it is necessary to hold the number of proposals to the lowest practical level. Thus, it is important to determine the number of proposals in a SIP that when increased will have little effect on the results but when decreased will have significant effect on the results.

Three test runs (five proposals, ten proposals, and 20 proposals per period) were made to examine the effects of the number of proposals per period on the variation in the values obtained for the capital growth rate. It is found that the variation in the values obtained for the capital growth rate is not appreciably reduced when more than 15 proposals are used. On the other hand, the variation of the capital growth increases noticeably when runs of five and ten proposals per period are made (more than 25%). Thus, by this process of elimination, it is believed that the fixed fifteen proposals per period would provide reliable results more efficiently.

6.3.5 The Number of Decision Times in a Study Period

The number of decision periods in a study period directly affects the simulation time, since numerous calculations must be made at each decision point. Therefore, it is necessary to utilize as short a study period as possible. However, it is also necessary to have sufficient decisions so that the effect of making decisions on a regular periodic basis would have time to stabilize and more tests of the criteria could be made. If the study period is too short, proposals with lives longer than the study period would not exert their full effect on the decision process. In the simulation, the number of decision times in a study period is set to $H = 20$, which is at least four times as great as the average life of the proposals in the SIP.

6.4 Applications of Decision Criteria to SIP and Computations of Statistics

6.4.1 Starting Conditions

Since this study is primarily concerned with the process of making capital budgeting decisions for a firm that is in full operation, special provision is made for generating cash receipts from hypothetical investments made prior to the initial decision point for the study period. The initial amount of dollars B_0 available at the first decision time (t = 0) is assumed to be composed entirely of receipts generated by previously implemented projects. Accordingly, the amount of funds B_1 available for investment at decision time t = 1 can be viewed as consisting of the cash receipts R_1 coming fue at t = 1 for those investments made before t = 0, plus those receipts coming due at t = 1 for those investment made at t = 0. In a similar manner, B_2 can be viewed as consisting of the receipts R_2 coming due from investments made prior to t = 0 and the receipts coming due from the investments made at t = 0 and t = 1. Therefore, as t increases, the contribution from those investments to the budget available at t (B_t) decreases, while the contribution from the investments made at t = 0 and later increases. Ultimately, all the investments made prior to t = 0 eventually will be terminated and their contribution to the budget will become zero. Thus, in the simulation, when the contribution from those investments made prior to t = 0 is minimum, the firm is said to have reached a steady-state condition.

In an attempt to approximate the steady-state conditions of an operating firm with the first few decision times of a simulation run, it is necessary to estimate the values R_t, where R_t is the cash receipts coming due at t for those investments made before t = 0. Thuesen [90, Chap. 6], in his simulation model, develops a simple methodology to derive a meaningful estimate of R_t as a function of t. To obtain this estimate of R_t with a reasonable amount of computation, he assumes that all investments made prior to t = 0 would be uniform series investments with a life equal to the average life of the proposals in the SIP's and with a rate of return equal to the average capital growth rate $\bar{g}$ of the firm. He also argues that even though the proposals appearing in the SIP's do not have this uniformity assumption, it should be noted that the combination of many proposals with different cash receipt patterns and different lives can result in a series cash flow that approximates a uniform series. Based on this assumption, he derives an equation to estimate B_t as a function of initial input parameters B_0, $\bar{g}$, and $\bar{n}$.

$$B_t = B_{t-1}(1 + \bar{g})$$

$$= B_0(1 + \bar{g})^t \tag{6-11}$$

where $t = 1,2,\ldots,\bar{n} - 1$,

$\bar{n}$ = the average life of investment proposals in the SIP's.

In this simulation, this methodology is directly applied to estimate B_t over the first phase of decision periods.

6.4.2 Cash Flow Realizations and Their Effect on Future Budget

As described in Section 6.4.1, the amount of funds (B_t) available for investment at decision time t is:

$$B_t = \sum_{j}^{N} \sum_{i=0}^{t-1} M_{jit} + R_t, \qquad t \geq 1 \tag{6-12}$$

and M_{jit} represents the receipt coming due at time t from the investment j made at decision time i. In order to determine the size of the budget at decision time t,(B_t), it is necessary to keep track of all realizations of cash flows associated with proposals undertaken before the current decision time t. Since all proposals are represented in a probability tree format, a particular cash flow occurring at time t is randomly selected from a uniform distribution for each proposal.

To visualize the simulation process, consider proposal j whose probability tree is shown in Figure 6-22. In Figure 6-22, the path 01 → 12 → 23 → 36 represents the actual realization of the cash flows associated with proposal j so that only this information is kept in the simulation as simulation progresses. However, it must be understood that at the time when the project is proposed, complete information about the probability tree is required to calculate the statistics for each decision criterion.

6.4.3 Selection of Proposals with Budget Limitations

All decision criteria discussed in Chapter V require the selection of proposals that maximize their objective function without allowing for the acceptance of fractional proposals. Under the assumption that proposals considered at each decision period are independent (see Section 5.2.4 and Section 5.3), the optimization technique to be used in the project selection is zero-one integer programming. However, the structure of the decision problem in this study is to maximize an objective function

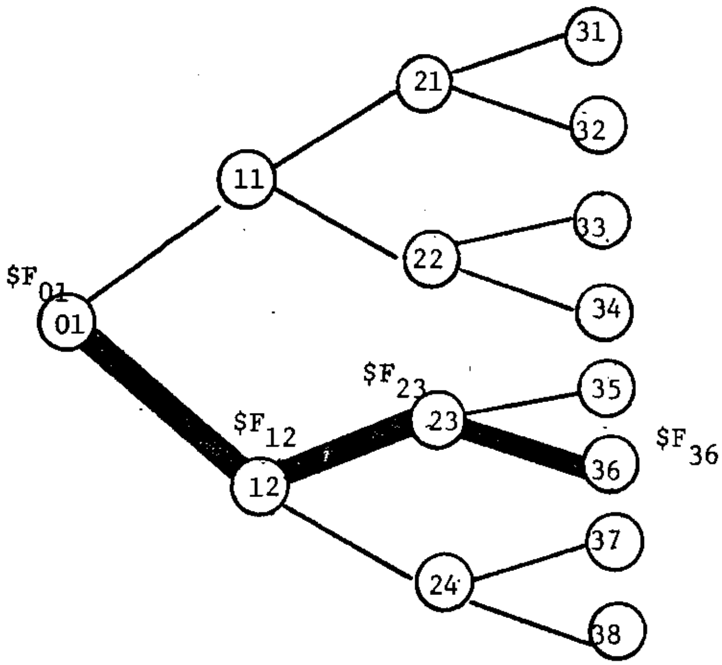


Figure 6-22. Time Path of Realizations of Cash Flows

with a single budget constraint, which results in a general knapsack problem. Therefore, a zero-one knapsack subroutine is utilized for the optimal selection of proposals for each decision period [103].

6.4.4 The Horizon Value as a Measure of Effectiveness

The basis for comparing the effectiveness of different decision criteria discussed in Chapter V is their ability to maximize the total accumulation of a firm's capital over a sequence of decisions. In this study, this accumulated capital or "horizon value" at a particular decision time $t = T$ is defined as the present worth at time T of the future receipts of investments that extend beyond T but were made on or before time T [90, Chap. 6]. Therefore, by definition, a proposal j with rate of return g_j undertaken at time T has an unrecovered capital of

$$S_j = \sum_{t=T+1}^{\infty} F_{jt}(1 + g_j)^{T-t} \tag{6-13}$$

Thus, in the simulation, each proposal j that is undertaken by the firm at a period t needs to be classified by its rate of return g_j to compute the total capital at decision time t.

Since each proposal is represented by a probability tree, g_j should be computed from the actual realization of a cash-flow path in the tree as shown in Figure 6-22. This requires accounting for all the sequence of cash-flow realizations for each proposal undertaken during the decision periods.

Once the respective rate of return for each proposal undertaken is identified, then the cash flows for all proposals with a certain rate of return g_j are combined as to amount and timing. As an example, suppose Proposals A, B and C are undertaken at t = 0, 1, 2, respectively. The cash flows and their rates of return g_j are given in Figure 6-23. The total capital at T = 2 would be

$$\begin{aligned}\text{Total Capital } (T = 2) = {} & [F_{A2} + F_{A3}(1 + g_A)^{-1}] \\ & + [F_{B1} + F_{B2}(1 + g_B)^{-1} + F_{B3}(1 + g_B)^{-2}] \\ & + [F_{C1}(1 + g_C)^{-1}]\end{aligned}$$

Thus, in general, the total capital at decision time t = T is defined as

$$\text{Total Capital at T (Horizon Value)} = \sum_j \sum_{t=T+1}^{\infty} F_{jt}(1 + g_j)^{T-t} \tag{6-14}$$

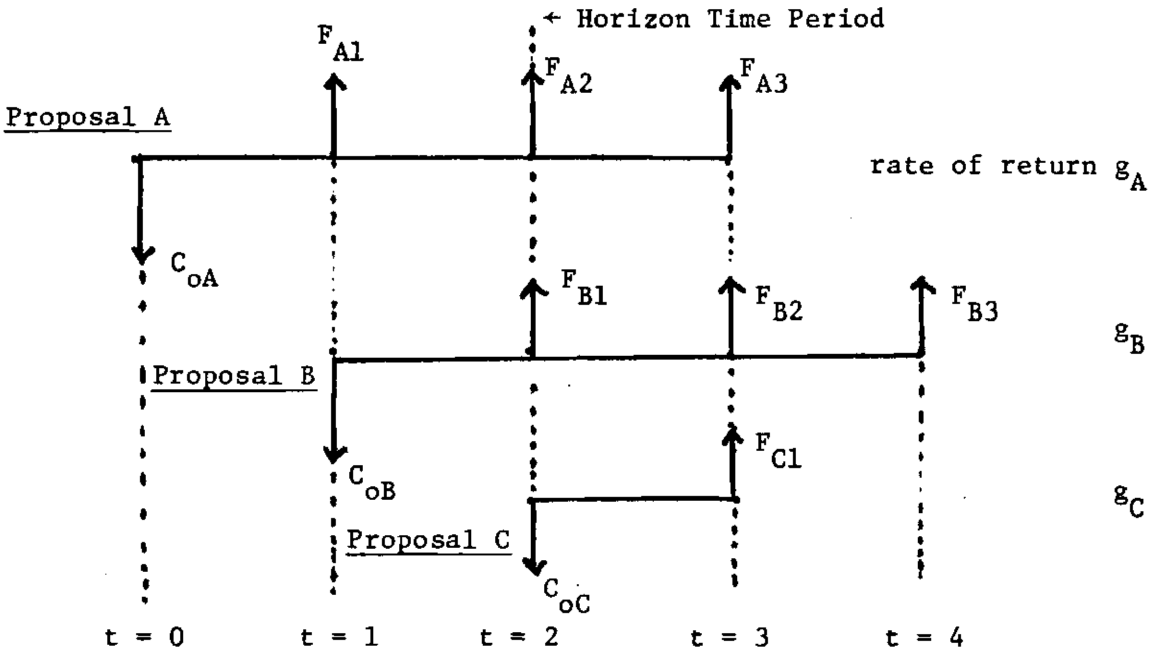


Figure 6-23. Illustration of the Total Capital Concept

6.4.5 Replication of Simulation Runs

In order to compare the effectiveness of one decision criterion with another, each of the criteria being considered is applied to the same set of proposals each decision period. A completion of one simulation run includes 20 decisions over 20 periods, and a completion of one simulation run produces a single value of total capital (horizon value) determined for each criterion being tested.

Since the amount of wealth (horizon value) accumulated in each simulation run is a random variable, several simulation runs must be made to compute the mean of these horizon values and the variability of the values about their mean. For the same investment setting, 10 runs are made in this study. The decision of how many runs should be made for a particular set of parameters can be determined by the variability that is observed in the total capital figure after some preliminary runs are made.

In general, as the number of sample sizes (simulation runs) increases, the variability that is observed in the total capital figure is expected to decrease [90, Chap. 6]. Therefore, preliminary runs are made at 20 simulation runs but no appreciable improvement (only 7% reduction in variability) is realized from these additional runs. On the other hand, when preliminary runs are made at five simulation runs, the increase in variability observed in the total capital figure amounts to 22%. Therefore, in this study, the sample size is fixed at 10 runs.

Then runs also are believed to be a reasonable compromise between limited computer time and the desirability of accuracy in most of the simulation results. The actual computer time required to complete 10 runs varies considerably, depending on the type of proposals being generated. When all the proposals have a single-payment type of cash flow, it takes approximately 1.20 minutes to generate data for 10 runs, around 6 minutes to compute the statistics required by each decision criterion, and around 25 minutes to solve 10 linear programming problems formulated by Weingartner's Horizon Model. On the other hand, when all proposals are a series-payment type of cash flow, it takes around 5 minutes to generate only input data for the 10 runs (3,000 proposals), approximately 8 minutes to compute all the statistics required by each decision criterion, and around 33 minutes to solve 10 linear programming problems formulated by the horizon model. These computer times do not include any input-output time required by the CDC Computing System at the Georgia Institute of Technology.

CHAPTER VII

THE SIMULATION RESULTS AND ANALYSIS

In this chapter, the simulation results produced by the models described in Chapter VI are presented and analyzed. To provide a background regarding the various simulations undertaken, the basic investment situations are described. In this study, three types of investment situations are described and their specific investment parameters are defined. The simulation results based on these investment settings are compared, and the performance of the PB criterion as compared with the decision criteria is discussed. To examine the effects of critical input parameters on the performance of each decision criterion, the sensitivity of the specific input parameters are analyzed and conclusions are presented.

7.1 Introduction

The primary questions to be answered by this study can be summarized as follows:

1. What improvement in proposal selection can be attained with better information regarding the manner in which the cash flow uncertainty is resolved over time?
2. Is the project balance (PB) criterion an effective decision criterion when the criterion is applied to the multi-stage capital budgeting decisions? How does the PB criterion perform with respect to the other three criteria discussed in Chapter V? Also of interest, how do the three criteria compare with each other when investment

decisions are made on a regular periodic basis? (Recall that the purpose of the inclusion of the expected utility criterion is not to draw a firm conclusion on the effectiveness of the criterion as compared to other criteria, but to provide the reader with some inference about the criterion (see Section 5.3.4).

3. How sensitive are these criteria to changes in the significant parameters associated with a regular periodic decision process?
4. How much can one improve his investment decisions with perfect knowledge of future investment opportunities?

To gain answers to these questions, it is necessary to apply decision criteria to different investment settings.

7.2 Types of Investment

7.2.1 Definition

As described in Chapter VI, there are various parameters that characterize the underlying framework of the investment decision process. Using different combinations of these parameters, it is possible to provide a variety of investment situations.

7.2.1.1 Future Investment Opportunities. Two distinctive future investment opportunities are defined for this study.

1. The Schedule of Future Investment Opportunities Remains Constant. This classification indicates that the distribution of investment opportunities with rate of return g_k remains constant through time. In other words, the fractions f_k and the rates of return g_k remain constant throughout the study period (see Section 6.3.1.1). In particular, the shape of the schedule of investment opportunities is

considered to be either exponential or linear, as shown in Figure 6-3 and 6-4.

2. The Schedule of Investment Opportunities. This classification indicates that the proportion of good investments at a higher growth rate (rate of return) is increasing as the number of decision periods increases. In other words, the firm is expected to be seeing better investment opportunities in the future. The shape of the distribution of the schedule of investment opportunities considered in this category is the one presented in Figure 6-5.

For each of the two investment situations described above, there are two classes of expected cash-flow patterns. They are defined as follows:

1. Heterogeneous. This classification indicates that there is a wide variety of expected cash flow patterns being considered at any decision period. Thus, the decision maker is faced with proposals whose expected cash-flow patterns are uniform series, single payment, and gradient series. Recall that all proposals are to be expressed in a probability tree format which is based on one of these expected cash-flow patterns. Thus, in this case the analysis considers diverse probability trees in terms of the magnitude and timing of cash flows.
2. Single Payment. This classification indicates that all proposals being analyzed are described by a single-payment probability tree (see Section 6.3.3).

An investment situation that is classified as single payment is utilized in the analysis to provide an investment setting because the cash flow of the proposal is realized in a single lump at the end of its life and no receipts from those proposals will be available for investment for the intervening periods. As an example, the commitment of funds to a single-payment proposal with a 10-year life may mean that some very lucrative proposals would be foregone during those intervening periods, due to the lack of cash to finance these desirable proposals.

The heterogeneous classification is designed to represent the types of proposals that many investment firms are expected to generate during normal operating conditions. Since it is believed that there is significant similarity between the actual decision situation and the heterogeneous type of proposals, conclusions from the resulting data will be of practical importance.

7.2.1.2 Investment Situations. The three types of investment firms are precisely defined in the following sections in this chapter. These situations will be referred to subsequently as Company A, Company B, and Company C.

Company A has a constant distribution of its schedule of investment opportunities such as shown in Figure 6-4, while Company B has a growing schedule of investment opportunities such as described in Figure 6-5.

Company C represents the investment situation where the variations in the investment parameters are relatively stable throughout the horizon time. Company C also assumes a constant growth in future investment opportunities like Company A, but the shape of the distribution of SIP used is the exponential shape as shown in Figure 6-3.

When compared with Company A and Company B, Company B generally has less good proposals available for investment at a high yield at each decision period. Furthermore, Company C differs from both Companies A and B in that the average first cost of the proposals is a smaller portion of the expected budget such that there will be reasonably sufficient funds available to finance the more desirable proposals in the SIP at each decision period.

Companies A, B, and C have both heterogeneous cash flows and single-payment cash flows. In other words, each company is evaluated under different assumptions of cash-flow mix. The investment situation where Company A has heterogeneous cash flows is defined as Case A-I, and where there are single-payment cash flows, it is defined as Case A-II. For Company B, the heterogeneous and single-payment cash flows are defined as Case B-I and Case B-II, respectively. Case C-I and Case C-II have similar interpretations for Company C.

The reason for evaluating each company in different investment settings is to provide a sharp contrast between the single-payment investment situation and the heterogeneous investment situation. The difference among these three investment situations is reflected in the next three sections, where the specific parameters for each of these firms are described.

7.2.2 Constant Investment Schedules (Company A)—Parameters

Case A-1--Heterogeneous Cash Flows

1. Distribution of investment opportunities with rate of return g_k (Figure 6-4)—linear shape
2. Discount rate (i) = 15%
3. Rate of growth applied to the distribution of first cost of proposals in the SIP per period ($\bar{g}$) = .25 (see Section 6.3.1.4 and Section 6.3.2.2)

4. Size of average investment per proposal at the beginning of simulation (see Section 6.3.2.2)

$\bar{C}_0 = \$15,000$

$a_1 = \$\ 6,000$ $\quad$ $c_1 = \$11,000$

$a_2 = \$10,000$ $\quad$ $c_2 = \$15,000$

$a_3 = \$14,000$ $\quad$ $c_3 = \$19,000$

5. $i_\delta = 6\%$ (see Section 5.2.1)

6. Size of external funds (see Section 6.4.1)

$B_0 = \$30,000$ $\quad$ $B_3 = \$16,064$

$B_1 = \$26,345$ $\quad$ $B_4 = \$\ 8,924$

$B_2 = \$21,775$

7. Proposal life (see Section 6.3.2.4)

$L_{min} = 2,\ \bar{n} = 5,\ L_{max} = 8$

8. Probability of a particular expected cash-flow pattern (see Section 6.3.2.5)

   a) Probability of a single-payment type cash flow

$$Q_1 = .20$$

   Probability of a series-payment type cash flow

$$Q_1' = 1 - Q_1 = .80$$

   b) If the proposal is a series-payment type cash flow:

   Probability of the cash flow being a combination of decreasing series

$$Q_2 = .60$$

   Probability of the cash flow being a combination of increasing series

$$Q_2' = 1 - Q_2 = .40$$

c) The probability distribution of f_R is uniform distribution with $\mu(0,1)$ (see Section 6.3.2.5)

9. Generation of probability tree (see Section 6.3.3)

a) The probability distribution of $P(\ell/k)_t$ (from node k to ℓ at decision time t)

$$\text{Uniform distribution with } \mu(0,1)$$

b) The probability distribution of the magnitude of cash flows at chance nodes occurring at period t

$$\text{Normal distribution with } \eta(E[F_t d]\sigma^2_{F_t})$$
$$\text{where } \sigma^2_{F_t} = (.15)E[F_t], \text{ i.e., } \beta = .15$$

10. Number of decision times in study period (H) = 20

11. Number of proposals per period (N) = 15

12. Ratio of expected funds available for investment to the expected total value of the proposals whose growth rates (rates of return) are greater than MARR in the SIP (see also Section 6.3.1.4)

$$\begin{aligned} \text{Ratio} &= [B_0]/[(\bar{C}_0)(N)(\text{Prob}(g_k \geq \text{MARR}))] \\ &= [\$30{,}000]/[(\$15{,}000)(15)(.70)] \\ &= .19 \end{aligned}$$

Case A-II—Single Payment

1 - 7. Same as Case A-I

8. a) The probability of a single-payment type cash flow

$$Q_1 = 1$$

8b - 12. Same as Case A-I

<u>7.2.3 Growing Investment Schedules (Company B)—Parameters</u>

Case B-I—Heterogeneous

1. Distribution of investment opportunities with rate of return g_k

(Figure 6-5)—linear shape

$$g_k(t) = .36(.30 - .01(t-1))[RN]$$

where RN represents random number from uniform distribution with $\mu(0,1)$.

3. Rate of growth applied to the distribution of first cost of proposals in the SIP per period ($\bar{g}(t)$) = .27 (see Section 6.3.1.4)

6. Size of external funds (see Section 6.4.1)

B_0 = \$30,000 B_3 = \$16,348

B_1 = \$26,484 B_4 = \$ 9,146

B_2 = \$22,019

Other parameters are the same as those of Case A-I.

Case B-II—Single Payment

8. a) The probability of a single-payment type of cash flow

$$Q_1 = 1$$

Other parameters are the same as those of Case B-I.

7.2.4 Constant and Stable Investment Opportunities (Company C)—Parameters

Case C-I—Heterogeneous Cash Flows

1. Distribution of investment opportunities with rate of return g_k (Figure 6-3)—exponential shape

2. Discount rate (MARR) = 10%

3. Rate of growth applied to the distribution of first cost of proposals in the SIP per period ($\bar{g}$) = .20 (see Section 6.3.1.4 and Section 6.3.2.2)

4. Size of average investment per proposal at the beginning of simulation (see Section 6.3.3.3)

$\bar{C}_0$ = \$8,000

a_1 = \$4,000 c_1 = \$ 7,000

a_2 = \$6,000 c_2 = \$ 8,000

a_3 = \$9,000 c_3 = \$12,000

12. Ratio = .385 (see Section 7.2.2)

Other parameters are the same as those of Case A-I.

Case C-II—Single Payment

8. a) The probability of a single-payment type of cash flow

$$Q_1 = 1$$

Other parameters are the same as those of Case C-I.

7.3 Measures of Performance

7.3.1 Risk-Return Analysis

7.3.1.1 Efficiency Concept. By simulating the financial results of investments selected on the basis of a particular policy (use of a particular value λ or δ), the expected horizon value--along with the standard deviation of the horizon values obtained with that policy--will indicate the "efficiency" of the investment project set selected under that policy [62]. The expected horizon value and the standard deviation can be plotted to show the outcome of implementing a particular policy. Given the results of applying various policies it is possible to identify the policies that yield the maximum expected value for a given standard deviation and also yield the minimum standard deviation for a given expected value. Connecting the points associated with these policies defines the "efficiency frontier" [40].

It must be recognized that the efficiency frontier obtained through the procedures above is stochastic in nature. This is because the computations of the expected horizon values and the standard deviations about these expected values are based on a fixed sample. Thus, as the sample size changes, the efficient frontier is likely to change. Based on the information on the sample size, one can determine the confidence limit of the efficient frontier (see Figure 7-1).

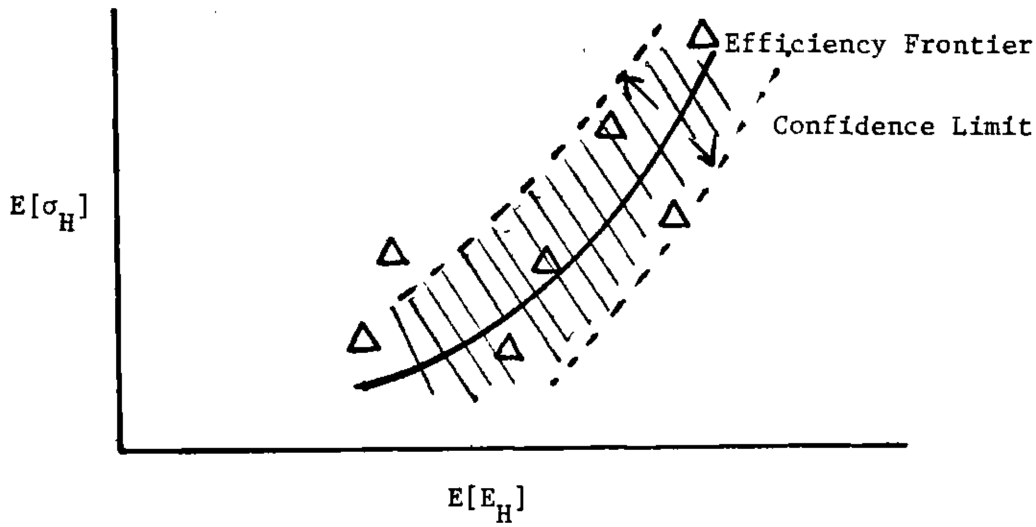


Figure 7-1. Concept of Efficiency Frontier

7.3.1.2 Risk Aversion Parameters. As discussed in Chapter V, a direct comparison of the PB criterion with other decision criteria calls for specification of a coefficient of risk aversion (δ) in advance. The same argument applies to the utilization of the mean-variance criterion in which a coefficient of risk aversion (λ) needs to be specified. Since different horizon values are possible for different values of δ (or λ and k), it is desirable to define the efficient set of investments by varying the coefficient of risk aversion, while holding all other parameters fixed [24]. This is because parametric variation of δ (or λ and k) allows the (E_H, σ_H) set to be traced (where E_H = expected horizon value,

σ_H = standard deviation about E_H).

In the simulation, for a given value of δ (or λ), 10 runs are performed. Then E_H and σ_H are computed from these 10 sample runs. For a different value of δ (or λ), another 10 runs are made, using the same parameters to compute E_H and σ_H. This procedure is repeated a number of times and the values of E_H and σ_H are plotted with $E[E_H]$ on the horizontal axis and $E[\sigma_H]$ on the vertical axis. In the simulation, the values of δ usually range from 0 to 3.0 while the values of λ range from 0 to .0005. In the simulation, the same values of δ were assigned to the corresponding values of k.

7.3.1.3 Dominance. Suppose an application of decision criterion 1 generates an efficient frontier AB and decision criterion 2 generates an efficient frontier CD for a particular investment setting, as sh n in Figure 7-1. Then, efficiency frontier AB is said to be dominated by efficiency frontier CD. This dominance, in turn, implies that decision criterion 2 can generate a higher expected value E_H without increasing variability in E_H as compared to decision criterion 1. Or stated differently, decision criterion 2 maintains the same level of risk (variability) without giving up too much E_H as compared with decision criterion 1.

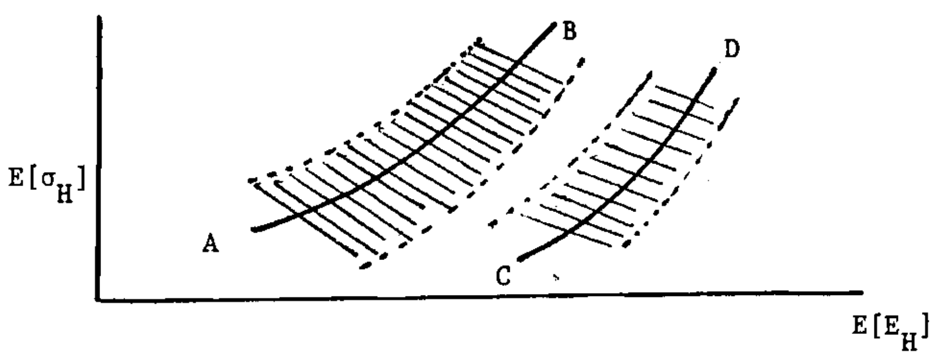


Figure 7-2. Illustration of the Dominance in Decision Criteria

7.3.2 Average Percentage of Total Capital Invested at i_δ

To gain an insight into the dynamics of the simulation model, it is of interest to understand the relationship between the budget amount and the projects undertaken over the decision periods. The average total capital invested at i_δ actually measures the funds remaining after investment; therefore, it is an indirect measure of the efficient use of available funds. In general, the primary reasons for funds being invested at this lower rate are:

1. There are not enough good proposals or they are too risky such that the decision criterion chooses to place the available funds in this risk-free investment i_δ.
2. There are no proposals with a small enough first cost to take advantage of the remaining funds.

In this simulation, the average percent of total invested i_δ (F_θ) is calculated in the following manner: First, it is necessary to calculate the total capital which is defined in Section 6.4.4. The total capital amount at each decision period consists of all capital available for investment plus the amount of capital that is due (discounted at respective rate of return) in future periods from investments made previously.

Then, for each decision time, a percentage figure is determined by dividing the amount of funds invested at i_δ at that decision time by the total capital at that decision time. Thus, in this study, F_θ represents the figure resulting from summing these percentages for each decision time and dividing by the number of decision times.

7.4 Simulation Results and Comparison of the Project Balance Criterion with Other Decision Criteria

In this section, the simulation results associated with the investment situations described in Section 7.2 are presented. The effectiveness of each decision criterion is compared with each of the other criteria under three distinctive investment situations (Company A, Company B, and Company C). For each company, two different cash-flow mixes in the SIP are assumed (heterogeneous vs. single payment), and the effectiveness of each decision criterion is compared with each of the other criteria under these different cash-flow mixes.

7.4.1 Constant Future Investment Opportunities--Company A

7.3.1.1 Heterogeneous Cash Flows (Case A-I). The simulation results for this investment situation are shown in Figure 7-3 (see detail statistics in Table A-1 in Appendix C). In Figure 7-3 the values of δ used for the PB criterion range from 0. to 5.0, while the values of λ used for the mean-variance (M-V) criterion range from 0. to .0005. Thus, the line connecting $\bigcirc^{1}$ and $\bigcirc^{7}$ represents the efficient frontier generated by the mean-variance criterion. The solid line running from ($\blacktriangle^{a}$) to ($\blacktriangle^{g}$) represents the efficient frontier generated by the project balance criterion for differing values of δ. The expected present worth maximization is a special case of the mean-variance criterion where the value of λ equals zero. The statistics obtained from the expected present worth criterion are shown as a shaded hexagon (⬢), and the statistics computed from the expected utility criterion are shown as a solid circle (●) in Figure 7-3.

For the project balance criterion, the greatest expected horizon

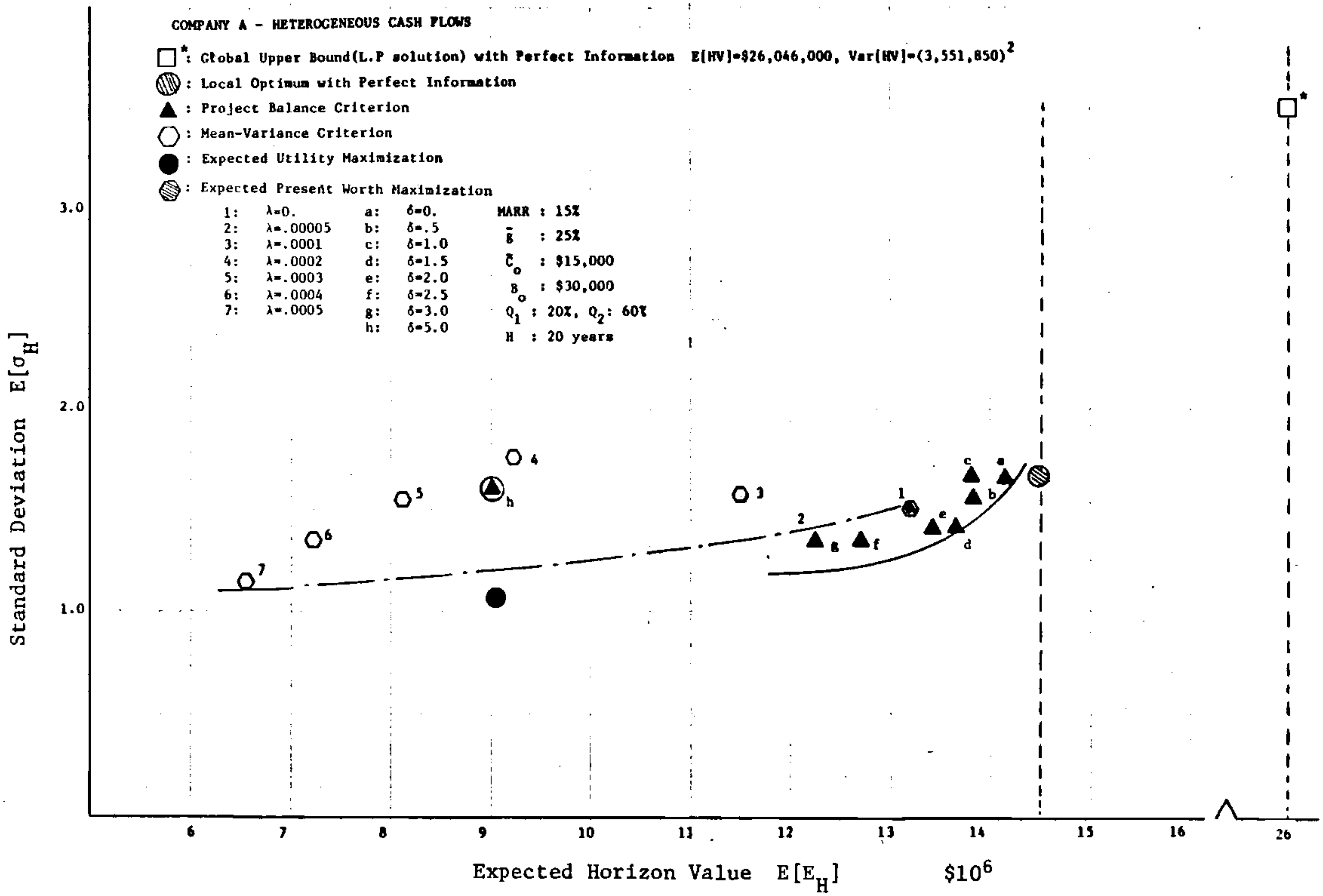


Figure 7-3. Risk-Return Chart: Company A--Heterogeneous

value (E_H = \$14,158 x 10^6) is obtained at $\delta = 0$, and as the value of δ increases, the value of E_H gradually decreases. This decreasing trend in the expected horizon value as δ becomes large can be explained as follows. As discussed in Chapter VI, the schedule of investment proposals submitted at each decision period contains a riskless proposal (highly liquid asset such as a bank account) whose growth rate is equivalent to i_δ. Such an investment yields a low return but has no variability of return from period to period. Assigning a higher value of δ implies that for a proposal, the requirement to meet the PB criterion becomes more restrictive (recall the fact that the value Z_i in Equation 5-4 must be positive in order to be considered for investment by the project balance criterion). Therefore, at a higher value of δ, there are not enough good proposals to meet this requirement so that the project balance criterion places the remaining funds in such a riskless asset for future investment in proposals generated in subsequent periods. Thus, for a higher value of δ, the firm generally can expect a smaller horizon value with less variability. Therefore, when δ goes to infinity, this criterion virtually requires the selection of proposals with zero variance. This is equivalent to saying that the firm eventually invests all its initial investment, including the externally supplied funds (B_t), in a riskless proposal (i_δ) over the study period. Then, theoretically the anticipated horizon value under this investment situation would asymptotically approach to an amount of \$304,004 with zero variance (\$304,004 = $B_0(1+i_\delta)^{20} + B_1(1+i_\delta)^{19} + B_2(1+i_\delta)^{18} + B_3(1+i_\delta)^{17} + B_4(1+i_\delta)^{16}$).

For the mean-variance criterion, the highest value of E_H occurs at $\lambda = 0$, which simply maximizes expected net present worth without reference to the dispersion of potential outcomes at each decision period. As the value of λ increases, the decreasing trend in the value of E_H is somewhat similar to the PB criterion. However, it is seen in Figure 7-3 that the firm using the M-V criterion cannot realize the same degree of risk which results in using of the PB criterion without incurring substantial loss in the value of E_H. As an example, when point 6 in Figure 7-3 obtained from the M-W criterion at λ = .0004 is compared with point g obtained from the PB criterion at δ = 3.0, they do not differ statistically from each other in terms of the variability in the expected horizon values (see Table A-1). However, the amount of E_H given up by using the M-V criterion rather than the PB criterion exceeds more than $5 million (75%).

It appears that the differences in the effectiveness of the PB criterion and the M-V criterion become negligible when the δ selected is sufficiently large. Consider point h (▲ h) in Figure 7-3. This point was computed at δ = 5 and very close to point 4 of the M-V criterion.

As indicated earlier in this section, for a higher value of δ or λ, there will be only a few proposals at each decision period which satisfy either criterion such that the size of funds available at each decision time becomes large enough to finance all these proposals. In other words, the ratio of funds available for investment to the total value of the proposals submitted for consideration per period increases as the value of δ or λ increases (see also F_θ figure in Table A-1). When this is the case, the effect of using different decision criteria becomes less pronounced.

However, it should be noted that it is rather unlikely for a rational decision maker to assign such a high value of (δ=5) to this investment setting, although δ theoretically can be chosen as ∞. For this reasoning, recall the fact that the magnitudes of cash flows associated with the probability trees were determined from normal distributions (see Section 6.3.3). This implies that the present worth of each proposal contained in the SIP submitted for consideration at each decision period can be assumed to be approximately normally distributed [43]. When this assumption is correct, the probability that the present worth of a proposal will fall below $5^{(\delta=5)}$ standard deviations from its mean value would be no more than (.287) x 10^{-7}! Thus, for a rational decision maker, the practical value of δ would be less than 3 in most cases (remember that there is only a 0.1 percent probability that it will ever fall below $E - 3_{\delta}$). For the practical ranges of value of δ from 0 to 3, it it seen that the efficient frontier generated by the PB criterion dominates the one obtained from applying the M-V criterion (statistically significant in terms of E_H with a degree of confidence of at least 95%).

The expected utility criterion yields the statistics represented by the solid circle in Figure 7-3. The point generated by this criterion has a relatively smaller expected horizon value and is virtually identical with point h of the PB criterion in terms of the expected horizon value. However its standard deviation is shown somewhat lower than any of the other points produced by the other decision criteria. This result was expected because the utility function utilized in this study has the property of diminishing marginal utility.

The property of this utility function indicates that for a risk-averse decision maker, his certainty monetary equivalent is always less than the expected monetary value for a given risk. Thus, when this criterion is compared with the expected present worth criterion, it is not expected to yield a higher expected horizon value than the one obtained from the expected present worth criterion.

Since the expected utility yields the lowest risk (lowest σ_H), it is of interest to determine if the variability of this criterion is statistically different from those of the other decision criteria. Assuming that the outcomes are normally distributed, a test to determine if the paired criteria differ with regard to their variability can be applied. The test used is the F-Test, and these tests indicated that the expected utility criterion is not statistically different from the other criteria with a degree of confidence of at least .95. In view of this fact, the smaller variability as compared to the reduction of expected horizon value does not seem to be a desirable trade-off.

As described in Section 5.3.4, if the decision maker has complete information about the SIP only when it is presented at each decision period, the local optimum with this perfect information is denoted by a shaded circle (⊘) in Figure 7-3. The expected horizon value and the standard deviation obtained from this local optimization are nearly identical with those of point a of the PB criterion (see Table A-1). This implies that when the project balance is used as a decision criterion under uncertainty, the criterion is rather effective because the improvement possible due to knowledge of the project realizations being considered in a single decision period is negligible.

Remember that the basic difference between the PB criterion and other decision criteria is that it recognizes the rate of uncertainty resolution. However, the value of uncertainty resolution was ascertained from the proposal's cash-flow patterns. Thus, the criterion places higher value on the proposals promising earlier and more highly probable realization of cash flows. In this sense, the value of knowing the project realizations is, to some extent, utilized in the PB criterion.

When the decision maker has complete information about all the future SIP's at the current decision time, Weingartner's horizon model generates the global upper bound at the point denoted by a rectangular (☐) in Figure 7-3. Again it should be noted that this solution was obtained from a linear programming formulation.

From Figure 7-3, it is seen that the value of knowing the project realizations for all decision periods is far more pronounced than in the case of the local optimum. This is largely because the perfect knowledge of present and future investment proposals allows for the optimization of sources of capital by keeping funds invested in highly liquid assets and carrying these funds from one period to the next to finance the attractive future proposals, rather than committing these funds to marginal investments at each decision time.

In Table A-1, F_θ represent the average percent of total capital invested at i_δ (see 7.3.2). It is also seen that the PB criterion generally has a smaller average percentage of the total capital invested at i_δ than does the M-V criterion. A reason for this is that the PB criterion in general favors projects having short lives and with most of their returns occurring early in their life. If the PB criterion selects a set of proposals in such a way at each decision period over the study period, there will be sufficient funds to invest in at least the best proposals in each SIP.

7.4.1.2 Single-Payment Cash Flows (Case A-II). As explained in

Section 7.2.1, the only difference between Case A-I and Case A-II is that Case A-II assumes all the proposals have single-payment cash flows. In comparison with the heterogeneous cash flows, this type of investment setting produces greater fluctuations in terms of funds flow over the study period (see Section 7.2). Although a firm is unlikely to be faced with this type of investment situation in actuality, this investment situation is designed to test the sensitivity of the PB criterion to an extreme investment situation.

The simulation results are shown in Figure 7-4 and the detailed statistics are tabulated in Table A-2. When Figure 7-4 is compared with Figure 7-3, it is seen that this investment setting provides a greater E_H along with a higher σ_H for all decision criteria. The maximum E_H for the PB criterion is obtained at $\delta = 1.5(d)$, and the efficient frontier curve of the PB criterion is even more positively down-sloped than the one generated by the M-V criterion. This implies that the rate of change of E_H with respect to $\sigma_H(\Delta E_H/\Delta\sigma_H)$ for the PB criterion is less than that for the M-V criterion. As seen in Figure 7-4, the PB criterion can achieve a higher E_H without incurring additional variability when all the points of the M-V criterion are compared with those of the PB criterion (statistically significant with a degree of confidence of at least .95).

Although it is dominated by the efficient sets of the M-V criterion, the performance of the expected utility criterion (in terms of E_H and σ_H) in relation to other decision criteria is not noticeably changed from that seen in Figure 7-3.

In this type of investment setting, the local optimum point is dominated by the efficient sets of the PB criterion. Remember that this

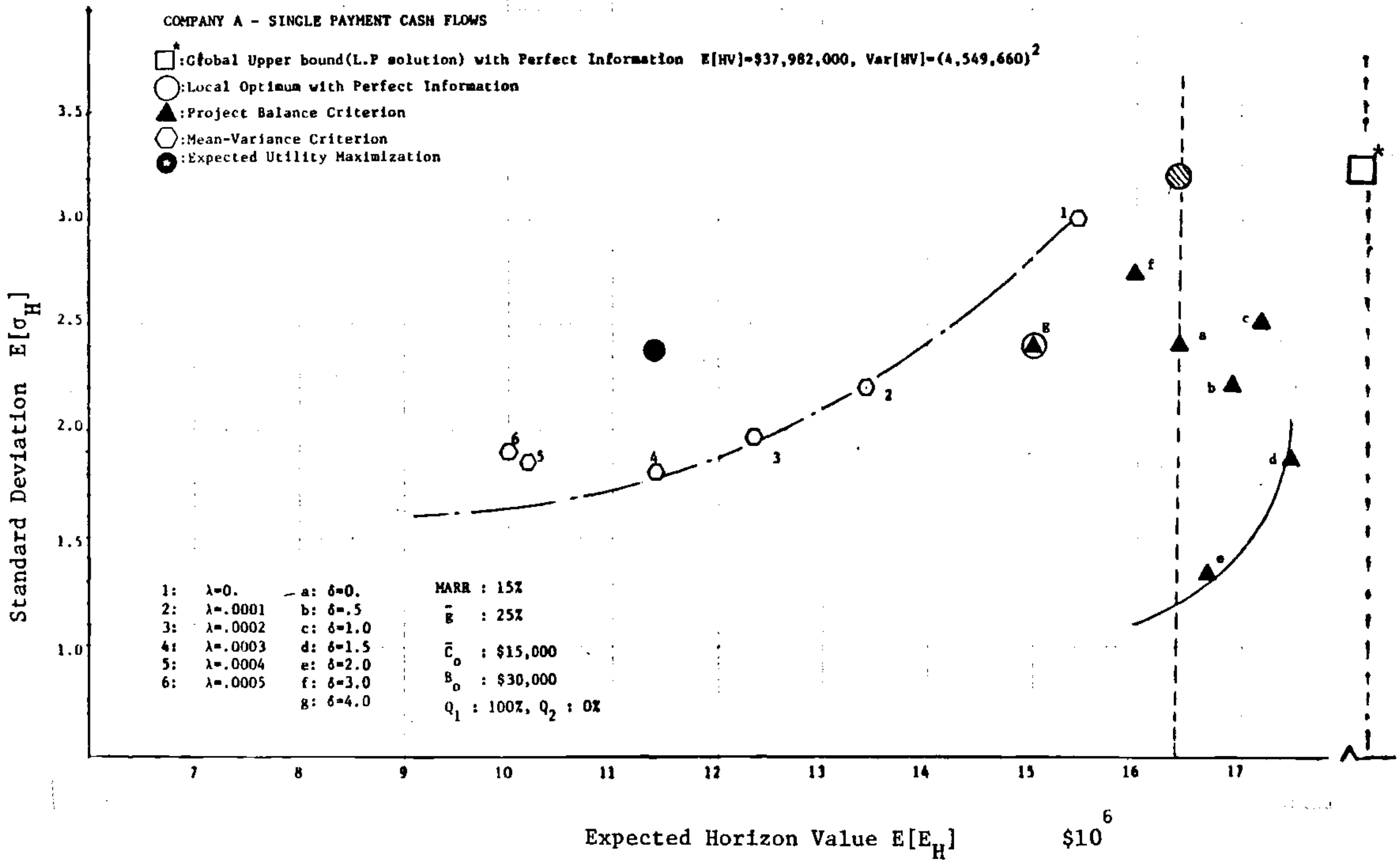


Figure 7-4. Risk-Return Chart: Company A--Single Payment

local optimum is obtained with the present worth maximization principle under assumed certainty (see Section 5.3.4.1). As discussed in Section 2.3.2.2, this present worth criterion focuses mainly on the terminal profitability. Suppose that the SIP submitted for a certain decision period contains two competing proposals, A1 and A2, both with a single-payment type of cash flows. Both have the same amount of outlays, but A1 has a one-year life, while A2 has a five-year life. Suppose both have the same positive expected net present worth, and the funds available for this decision period are just enough to finance either one of the two. If there is no future disbursement other than the initial investment, then the PB criterion will select proposal A1.

Now suppose perfect information about the realization of cash flows becomes available and the present worth of A2 is somewhat greater than that of A1. Then the present worth criterion will select A2, instead of A1. However, the commitment of funds to a single-payment proposal with a five-period life means that <u>no</u> receipts from that proposal will be available for investment for the intervening four periods. This may mean that some very lucrative proposals would be foregone during those four periods because of the lack of cash to finance these proposals. On the other hand, the PB criterion favors relatively short-term proposals such as A1 and keeps funds available for investment in attractive future proposals. Therefore, the selection of projects even under uncertainty could result in a greater total capital at the end of horizon time, as compared to the optimum selection of proposals that could be achieved with the <u>periodic</u> perfect information.

Finally, F_θ figures in Table A-2 provide a sharp contrast to the

F_θ figures in Table A-1. Observe that the PB criterion generally has a smaller average percentage of the total capital invested at i_δ than do the other decision criteria. The F_θ figures also indicate that as the investment situation moves from one of relative stability (heterogeneous cash flows) to a more variable situation (single-payment cash flows), the amount of total capital invested at i_δ increases substantially.

7.4.2 Growing Investment Schedules--Company B

Presented in this section are the simulation results from the utilization of data described in Section 7.2.3. Remember Company B differs from Company A only in that it has a growing distribution of schedule of investment opportunities as shown in Figure 6-5. Accordingly, the average capital growth rate ($\bar{g}$) of this firm would be different from that of Company A. This $\bar{g}$ is computed approximately at .27 (see Section 6.3.1.4). Otherwise, Company A and Company B are exactly identical in terms of other investment parameters.

7.4.2.1 Heterogeneous Cash Flows (Case B-I). The simulation results for each decision criterion are compared in Figure 7-5 and the corresponding statistics are tabulated in Table A-3. Figure 7-5 indicates that the PB criterion substantially outperforms all the other decision criteria (except the global upper bound) which are evaluated. In particular, the maximum value of E_H for the PB criterion occurs at $\delta = .5$ (point b in Figure 7-5), and this value is statistically significant over all other maximum values (E_H) achieved by other decision criteria with a degree of confidence of at least .95 (paired t-test).

In terms of risk, it is also seen that the PB criterion maintains a lower variability without substantial loss of E_H. When the PB criterion

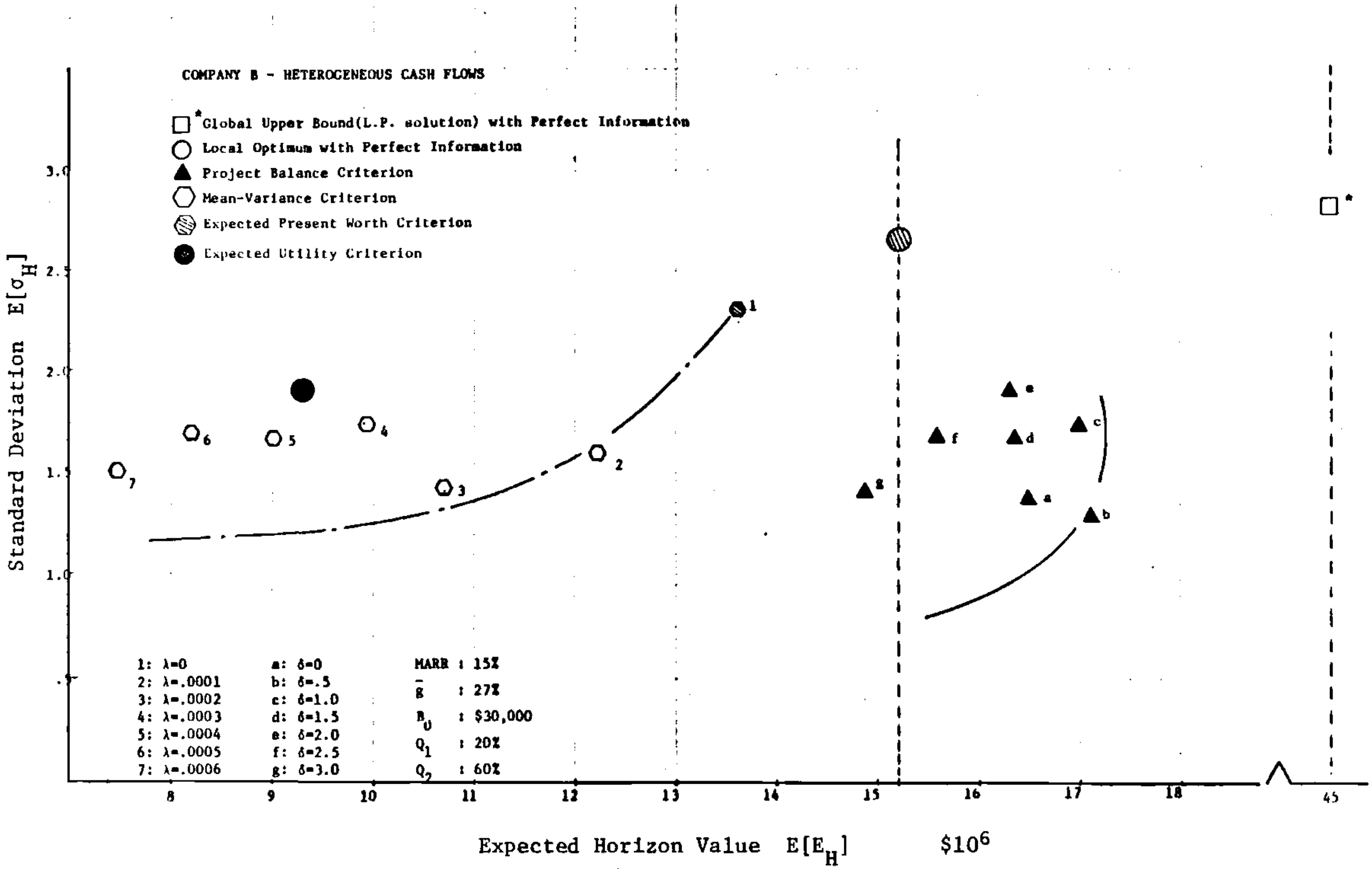


Figure 7-5. Risk-Return Chart: Company B--Heterogeneous

is compared with the M-V criterion and the expected utility criterion, the PB criterion produce substantially higher E_H than does any other criterion for any given level of risk.

As explained in Section 7.4.1.1, the reason that the PB criterion performs so well for this type of investment setting is that the PB criterion will select the proposals which promise earlier and highly probable realization of cash flows with relatively uniform marginal returns, instead of selecting some other proposals whose expecged returns are somewhat greater but where uncertainty about realization of cash flows persists longer than usual. Stated differently, an important feature of the PB criterion is its recognition of the advantage of keeping funds available for investment (greater flexibility) in attractive proposals which are available in the future. In addition, the investment situation is such that the firm expects to have an ever increasing proportion of good proposals available for investment at high growth rates throughout the study period. In this investment setting, the repeated application of the PB criterion at each decision period will provide reasonably steady funds to invest in at least the best proposals in each SIP. This is why the performance of the PB criterion becomes more pronounced under this investment setting as compared with the previous investment setting (Company A).

The E_H from the global upper bound solution for this type of investment setting is found to be almost 73% higher in value than the one that could be achieved under the previous investment setting. This dramatic increase in E_H value is due largely to the fact that the identification and complete knowledge of good investment prospects in later time periods results in transfer of more funds from period to period by investing in highly liquid assets. Thus, the value of information about future investment opportunities becomes more pronounced.

7.4.2.2 Single-Payment Cash Flows (Case B-II). The simulation results for Case B-II are shown in Figure 7-6 and the detailed statistics are tabulated in Table A-4. As learned from the analysis of Case A-II and Case B-I, it is expected that the performance of the PB criterion would be more successful for this type of extreme investment situation. Examination of Figure 7-6 makes it clear that the PB criterion consistently leads to selection of proposals that result in greater future value at the horizon time (E_H) than for any of the other three decision criteria. It is also statistically confirmed that the project selection by teh PB criterion also results in greater future value at the horizon time than the local optimum with perfect information.

The F_θ figures in Table A-4 further illustrate the interesting dynamics of this investment situation. Observe that the PB criterion generally has an average of 12% of the total capital invested at i_δ, while other decision criteria have an average of 27% of the total capital invested at i_δ. A higher F_θ value for the other decision criteria is due mainly to the fact that the repeated application of these decision criteria generally does not result in proposals having a first cost small enough to take advantage of the remaining funds at each decision period.

7.4.3 Constant and Stable Investment Opportunities (Company C)

As mentioned in Section 7.2.1, Company C differs from Company A in that

1) The type of the distribution of SIP is an exponential shape,

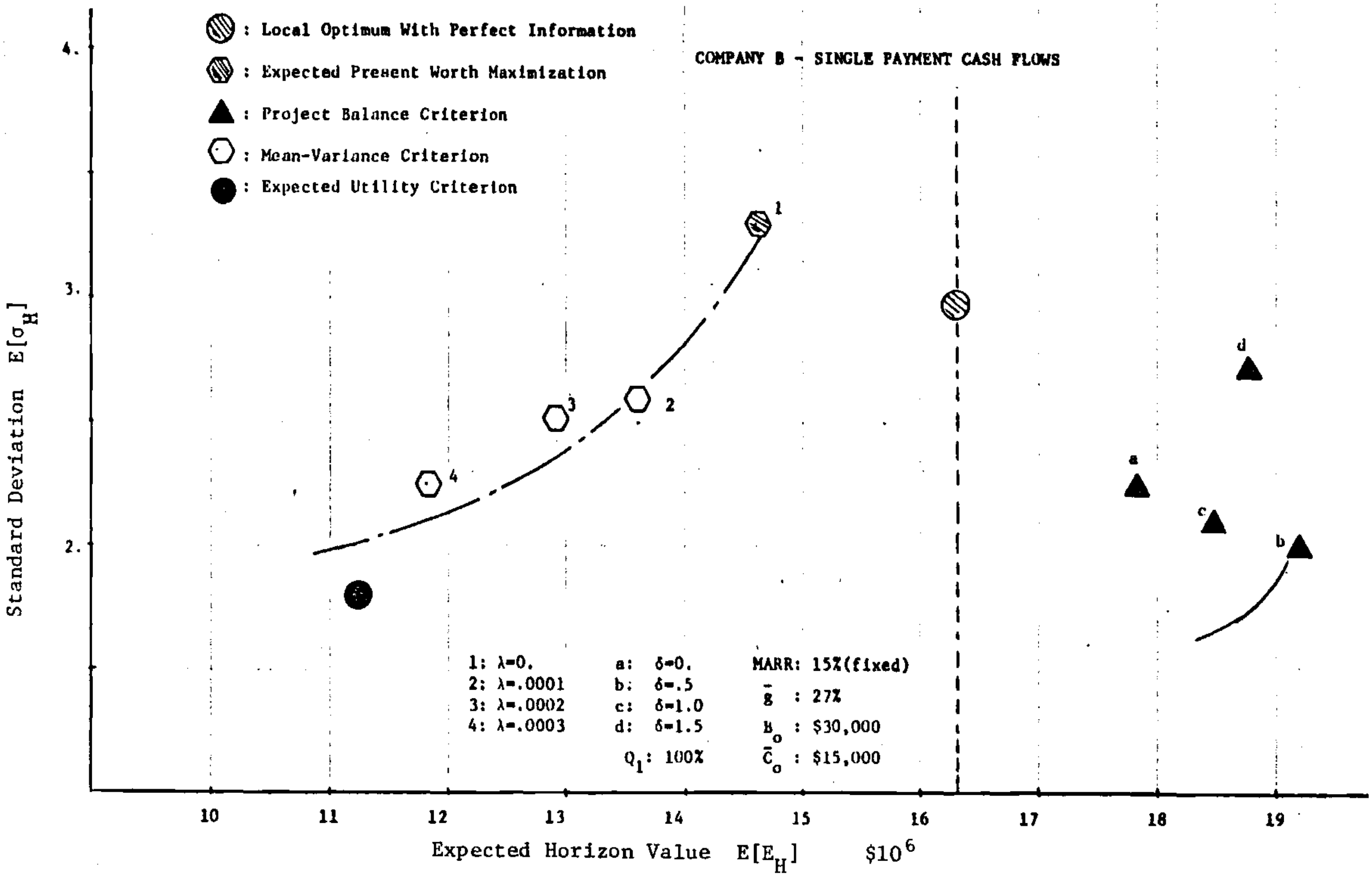


Figure 7-6. Risk-Return Chart: Company B--Single Payment

2) The discount rate used is 10% such that the firm is expected to grow an average of 20% ($\bar{g}$), and

3) The distribution of the first cost of proposals has $\bar{C}_0$ = \$8,000 (as compared to $\bar{C}_0$ = \$15,000 for Company A) such that the average first cost of the proposals is a smaller portion of the expected budget.

7.4.3.1 Heterogeneous Cash Flows. The simulation results for the utilization of data set forth in Section 7.2.4 for the heterogeneous case are presented in Figure 7.7, and the detailed statistics are summarized in Table A-5. Figure 7.7 indicates that the maximum E_H for the PB criterion occurs at δ = .8(\$4,810 × 10^6) and the maximum E_H for the M-V criterion occurs at λ = 0 with \$4,102 × 10^6, which is a relatively small difference as compared with the case in Company A. When point 1 of the M-V criterion (the expected or present criterion) is compared with point e of the PB criterion, the variability of point 1 is statistically significant at a degree of confidence of .90 (this was confirmed by an F test). In general, the efficient frontier of the PB criterion dominates the one generated by the M-V criterion.

The performance of the expected utility criterion appears to be as effective as the M-V criterion for this relatively stable investment setting. Point C of the PB criterion is close to the local optimum and virtually identical in terms of magnitude of E_H and σ_H. It is also seen from Figure 7-3 that the PB criterion results in an expected horizon value close to the local optimum with selection of appropriate δ value. In this relatively stable investment situation, the value of having complete information for the future investment opportunities is substantially reduced as compared with that experienced by Company A. The difference in the

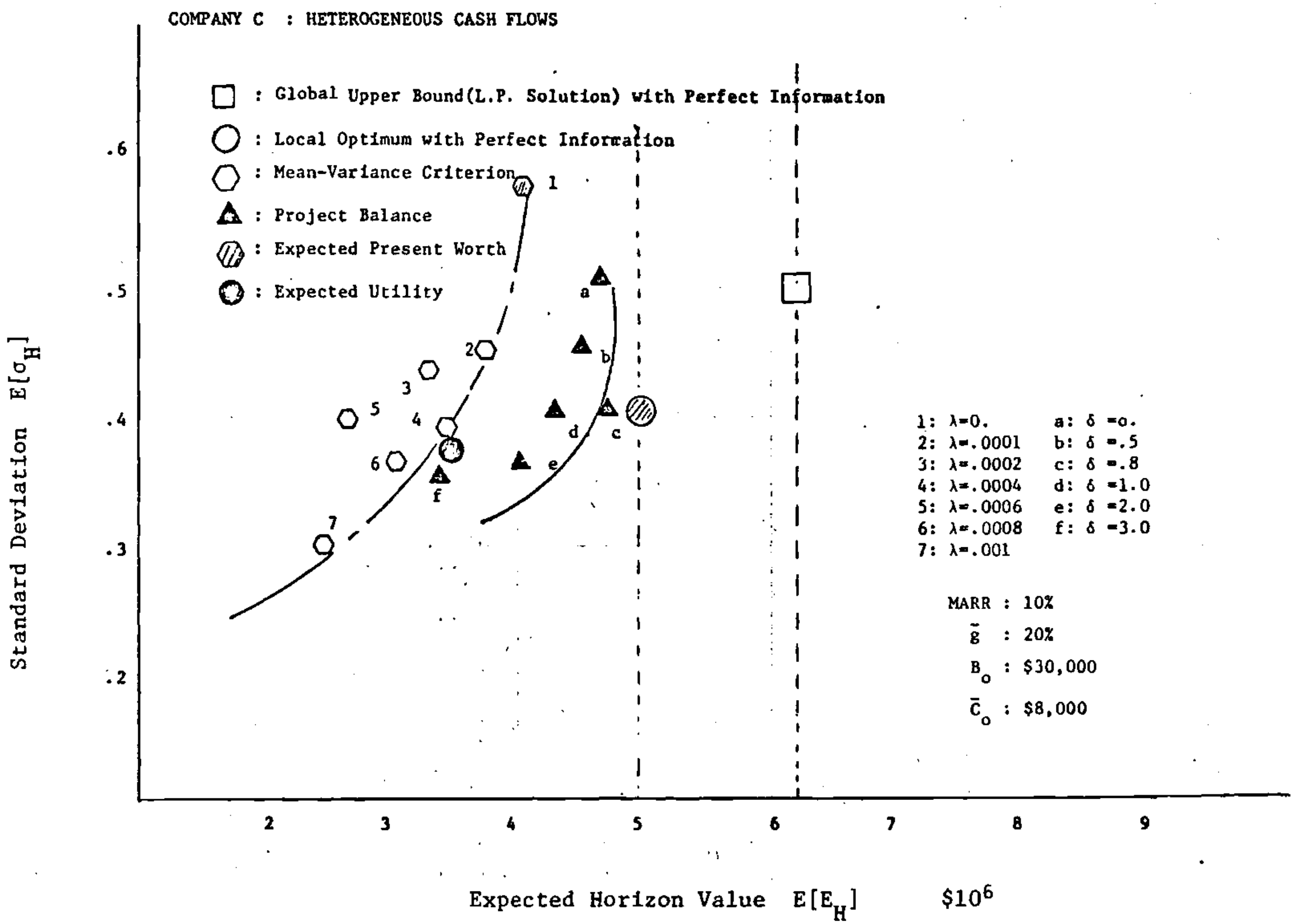


Figure 7-7. Risk-Return Chart: Company C--Heterogeneous

expected horizon value between the local optimum and the global upper bound is only 1.2×10^6, which is about 25% of the local optimum. The relative difference between the local optimum and the global upper bound for Company A is 79%.

7.4.3.2 Single-Payment Cash Flows. For the single-payment case, the performance of each criterion is displayed in Figure 7-8. Figure 7-8 shows that the relative effectiveness of the PB criterion over the other decision criteria is consistently superior in terms of both maximizing E_H and maintaining lower variability σ_H.

For the single-payment type of cash flow, it appears that the relative value of having perfect information about the future investment opportunities is greater than in the heterogeneous case. (Compare Figure 7-3 with Figure 7-4, and also compare Figure 7-7 with Figure 7-8.) The reason for this may be explained as follows. Since all proposals have a single-payment type of cash flow, the consequence of selecting Proposal A rather than Proposal B at some decision time t may be that the entire sequence of subsequent decisions is different than it would have been if Proposal B had been chosen. That implies that the value of having perfect information increases.

7.4.4 Effects of the Discount Rate (MARR)

Since all the decision criteria discussed in Chapter V are a function of the discount rate, it is of interest to examine the sensitivity of the effectiveness of each decision criterion to change in the discount rate used. Two types of sensitivity analysis are utilized in this study.

In the previous two sections, it was assumed that once the firm selects a discount rate for the evaluation of proposals at the very

COMPANY C : SINGLE PAYMENT CASH FLOWS

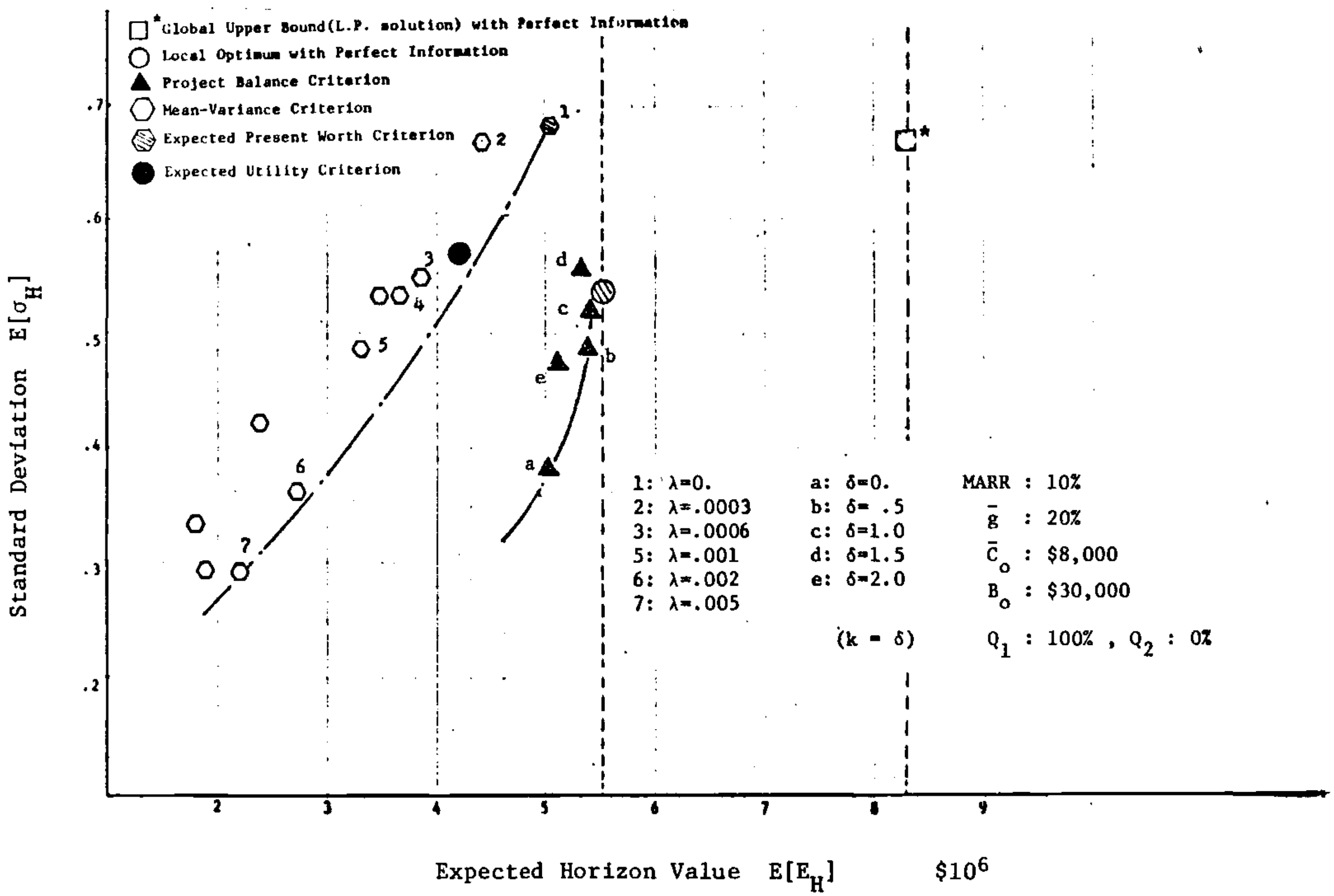


Figure 7-8. Risk-Return Chart: Company C--Single Payment

beginning of the decision period, the same discount rate is applied to the evaluation of proposals submitted throughout the study period. The first approach is to investigate how sensitive the effectiveness of each decision criterion is for various discount rates.

For Company B, remember that the firm has growing investment opportunities over the study period. As defined in Chapter VI, the growing investment opportunities exhibit an increasing proportion of good proposals available for investment at a higher growth rate (rate of return) as time passes. This situation indicates that as the length of time increases, the number of investment opportunities that yield at least the MARR (which remains constant) also increases. However, in the capital rationing context, if the MARR selecged is relatively low with respect to the number of productive investment opportunities available, this rate may allow the acceptance of a large number of proposals, some of which are marginally productive. Therefore, it may be reasonable for a firm to consider adjusting the discount rate with regard to the investment opportunities available at each decision period. Thus, the second analysis is to examine the effectiveness of each decision with the time-variant discount rate over the study period.

7.4.4.1 Effects of a Time-Invariant Discount Rate (for Constant Future Investment Opportunities). For Company A (see Figure 7-3), the discount rate used is 15%. Five different discount rates ranging from 6% to 30% are selected to compute the statistics (E_H and σ_H) associated with each decision criterion. Since the PB criterion and the M-V criterion require that the corresponding coefficient of risk aversion (δ = k, λ) be specified, the coefficient of risk aversion used is δ = .5 for the PB criterion and λ = .0001 for the M-V criterion.

The statistics obtained from varying the discount rate for each decision criterion for Case A-I are summarized in Table 7-1. On the other hand, Figure 7-9 illustrates the relative advantage (in terms of expected horizon value) of using the PB criterion over the other decision criteria as a function of the discount rate. Accordingly, Figure 7-10 depicts the standard deviations (σ_H) and E_H associated with each decision criterion as a function of the discount rate.

Table 7-1 and Figure 7-9 indicate that all decision criteria except the M-V criterion produce the maximum E_H at a discount rate of 20%. The maximum E_H value for the M-V criterion occurs at a discount rate of 15%. These maximum values (E_H) support the argument that there is an optimal discount rate for this capital rationing process. As the discount rate varies from 6% to 15%, the E_H value produced by each decision criterion increases and the M-V criterion alone reaches its maximum value E_H. At a discount rate of 20%, all other decision criteria reach their maximum value E_H.

At a higher discount rate, more than 20%, the values E_H for all the criteria decrease, and at 30%, there is a substantial reduction in the value of E_H. The argument explaining these changes in E_H values is that if the discount rate selected is too high, many proposals that have good returns are rejected. A lower discount rate allows the acceptance of a large number of proposals, some of which are marginally productive, and this results in an E_H value lower than the value that could be achieved if the optimal discount rate were used. Thus there is a trade-off between being too selective or not being selective enough. It must be noted that

Table 7-1. Effects of a Time-Invariant Discount Rate--Company A

Decision Rule \ MARR	6%	10%	15%	20%	25%	30%
Global L.P. Upper Bound (Horizon Model)	26,046 (3,552)	26,046 (3,552)	26,046 (3,552)	26,046 (3,552)	26,046 (3,552)	26,046 (3,552)
Local Optimum	11,633 (1,928)	12,967 (1,580)	14,524 (1,658)	15,986 (1,570)	15,410 (1,195)	11,182 (2,399)
Project Balance (at δ = .5)	14,311 (1,430)	14,099 (1,454)	13,856 (1,514)	14,233 * (1,543)	13,373 (1,288)	9,875 (1,799)
Expected PW Max	10,135 (2,128)	11,846 (1,931)	13,299 (1,510)	14,549 (1,466)	14,022 (1,316)	10,353 (1,941)
Mean-Variance (at λ =.0001)	9,293 (1,878)	10,845 (1,812)	11,438 (1,552)	11,113 (1,979)	8,925 (1,791)	5,528 (1,580)
Expected Utility (U(Z)= log(Z))	6,193 (1,272)	6,964 (1,358)	9,028 (1,048)	11,323 (1,388)	10,787 (1,998)	6,014 (2,428)

* At a discount rate of 20%, the maximum E_H for the PB criterion occurs at δ=0. ( ) : σ_H
The values of E_H and σ_H at δ=0 are E_H = \$14,750, and σ_H = \$ 1,455.

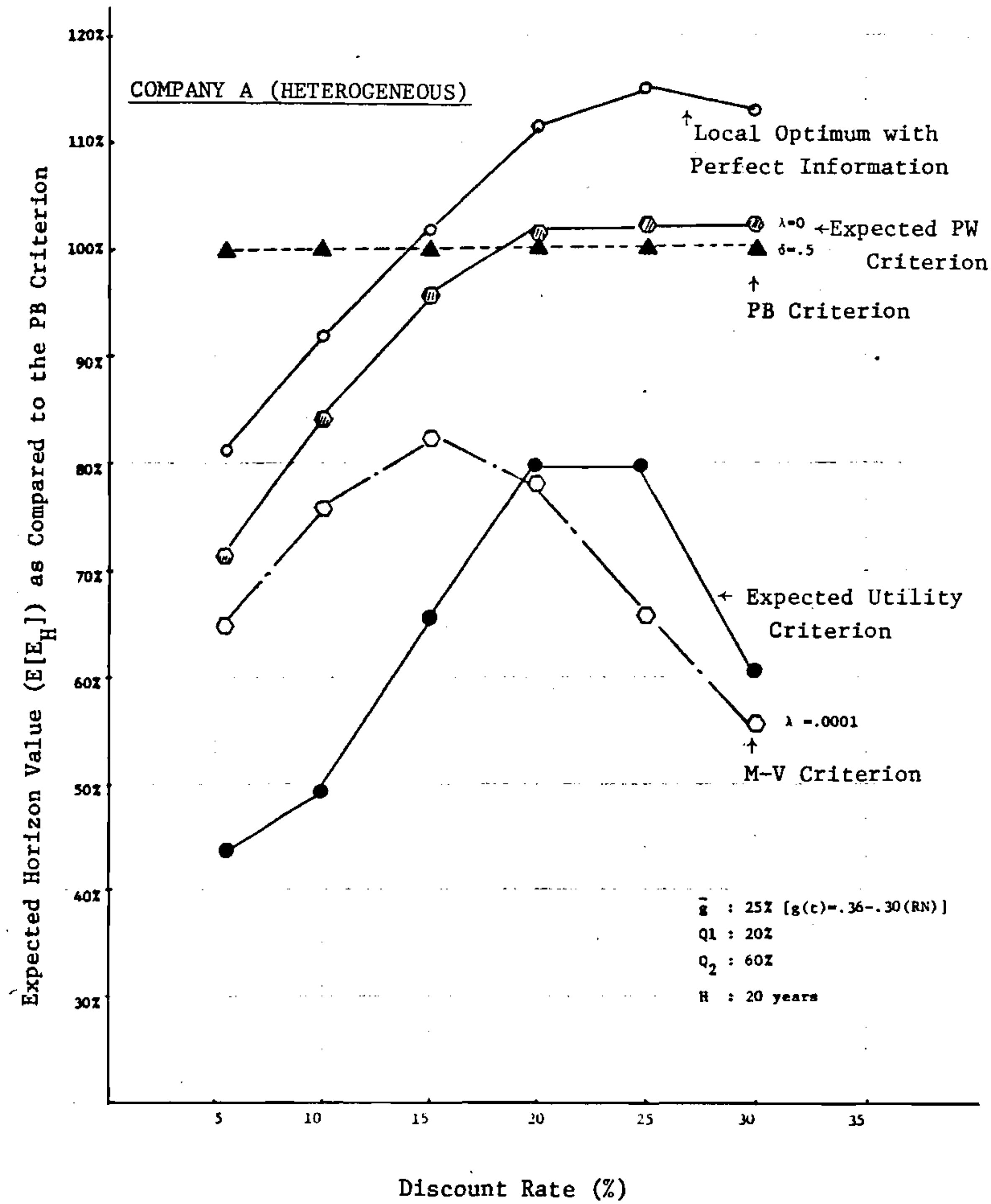


Figure 7-9. The Relative Advantage of Using the PB Criterion over the Other Decision Criteria as a Function of the Discount Rate

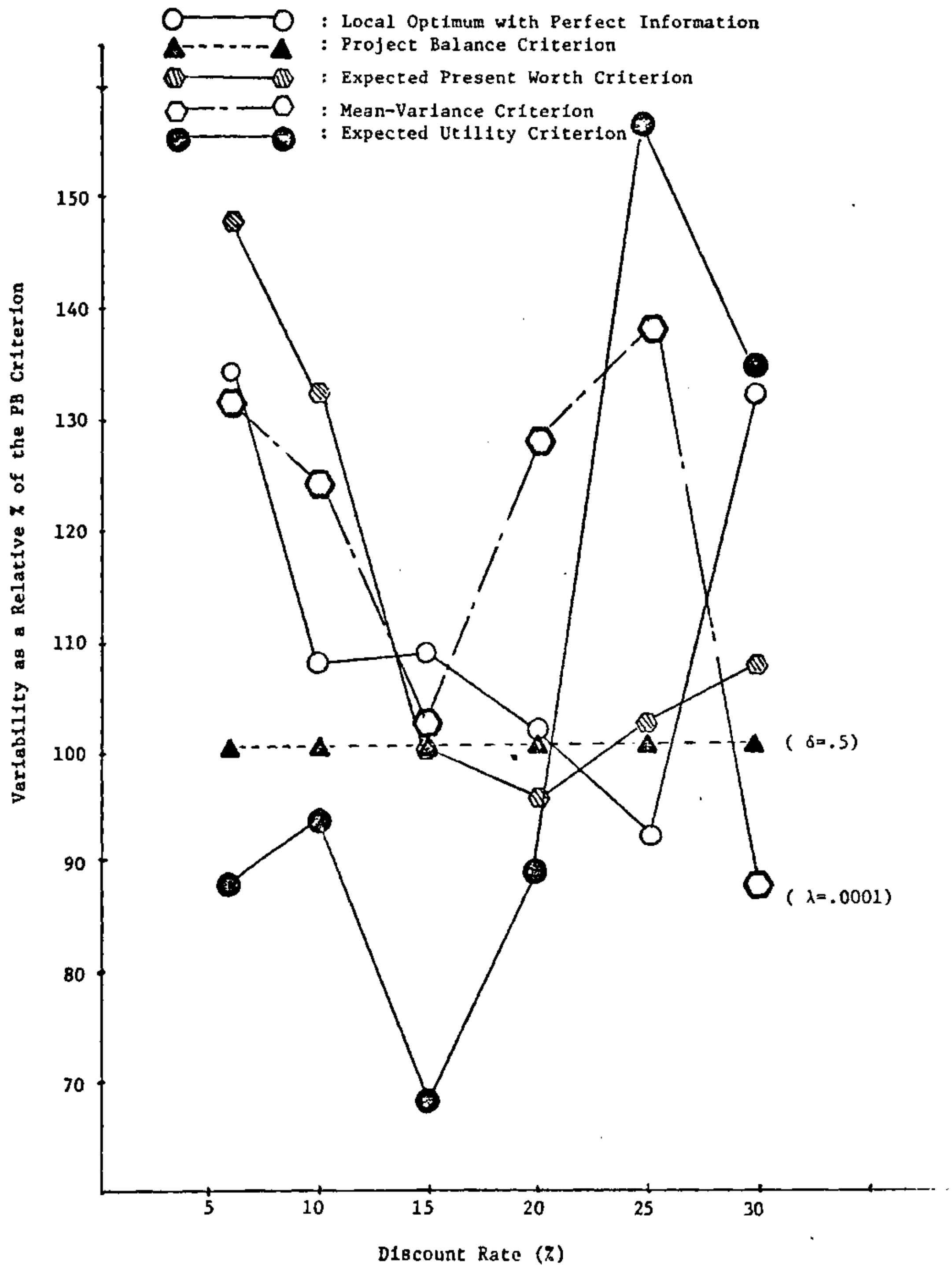


Figure 7-10. Variability in E_H as a Function of the Discount Rate Selected

the global upper bound does not change for different values of the discount rate. This is because Weingartner's horizon model is not a function of a discount rate (see Section 5.3.5.2).

Careful examination of Table 7-1 reveals the fact that the effectiveness of the PB criterion is not highly sensitive to changes in the discount rate selected. In Table 7-1, compare the E_H value obtained at a discount rate of 6% with the one at a discount rate of 20% for the PB criterion. They are virtually identical, while the changes in E_H values over this range of discount rates for the local optimum, the expected present worth criterion, the M-V criterion, and the expected utility criterion are 37%, 43%, 19.6%, and 82%, respectively. It appears that the expected utility criterion is most sensitive to changes in the discount rate. In terms of absolute magnitude of the E_H value, the PB criterion also produces the largest maximum E_H value among other decison criteria under uncertainty. In Table 7-1 it appears that at a discount rate of 20%, the expected present worth criterion seems to produce the largest maximum horizon value ($14,549). However, at that same discount rate, the maximum E_H for the PB criterion occurs at $\delta = 0$, with E_H = $14,750, which is greater than the E_H value obtained from the expected present worth criterion.

From Figure 7-9, it is seen that the performance of the PB criterion eventually approaches the expected present worth criterion at a discount rate higher than 25%. The reason for this trend is that both criteria accept only those proposals with growth rates (rates of return) greater than or equal to its particular discount rate (MARR) value. Thus the number of proposals that are acceptable diminishes as the discount rate becomes greater. However, as the number of proposals that are acceptable decreases, the ratio of expected funds available for investment to the expected total value of the proposals in the SIP increases. Eventually there will be sufficient funds to finance all the reduced number of proposals. As an example, from the distribution of SIP shown in Figure 6-4, it can be estimated that only 36% of total investments could be invested at or above a growth rate of 25% and 20% of total investments could be invested at or above a growth rate of 30%. This, in turn, implies that only about five proposals out of 15 generated at each devision period would be considered for investment at MARR = 25%, while only three proposals out of 15 would be considered for investment if MARR = 30%. On the other hand, the ratios of expected funds available for investment for a MARR = 25% and 30% would increase from 19% (at MARR = 15%) to 40% and 67%, respectively (see Section 7.2.2). Therefore, there will be a small number of proposals with sufficient funds to finance them and the effect of using different criteria becomes less pronounced.

Figure 7-10 indicates that the PB criterion also maintains a lower variability (σ_H) than other decision criteria. The expected utility criterion maintains lower variability than does the PB criterion for the range of MARR from 6% to 20%. However, at a discount rate higher than 20%, it is seen that the performance of the expected utility criterion drops substantially and eventually maintains the highest variability when compared with the other decision criteria. It should be understood that the lower risk of the expected utility criterion over the range of a lower discount rate cannot lead to the conclusion that the performance of the expected utility criterion is better than that of the PB criterion. When the expected utility criterion is compared with the PB criterion, it is observed from Figures 7-9 and 7-10 that a 12% relative reduction in risk for the expected utility criterion at a MARR = 6% results in a 56% relative reduction in E_H value and a 31% relative reduction in risk of the expected utility criterion would require a 35% relative reduction in E_H value; finally at MARR = 20%, a 11% reduction in risk would result in a 20% relative reduction in E_H. Thus, in absolute dollar terms, the use of the PB criterion eventually leads to improved wealth accumulation.

7.4.4.2 Effects of a Time-Variant Discount Rate. As discussed in Section 7.4.4.1, the changes in discount rate would directly affect the expected number of proposals that are acceptable at each decision period such that the ratio of expected funds available for investment also changes (see Section 7.2.2). Thus, in order to eliminate this effect of budget and number of proposals available at each decision period, this ratio is held constant over the planning horizon.

It is seen from Figure 6-5 that if the MARR is 15%, then 70% of the total investments could be invested at or above a rate of return of 15%. This, in turn, implies that approximately 10 proposals out of 15 at each decision period would have an expected present worth greater than or equal to zero. Thus, a discount rate $\bar{m}(t)$ which causes the ratio to remain constant over the planning horizon when the investment opportunities are increasing can be found as follows:

For growing investment opportunities, from Equation 6-2 in Section 6.3.1.3,

$$g_k(t/F_k) = .36 - (.30 - .01(t-1))F_k$$

$$t = 1, 2, \ldots , H$$

By letting F_k = .70 and MARR = $\bar{m}$, the discount rate to be used at each decision period is

$$\bar{m}(t) = .36 - (.30 - .01(t-1))(.70)$$
$$t = 1, 2, \ldots, H$$

Thus, at t = 1, $\bar{m}(1)$ = 15%

t = 10, $\bar{m}(10)$ = 21.3%

t = 2-, $\bar{m}(20)$ = 28.3%

This time path of $\bar{m}(t)$ is represented by line EF in Figure 6-5.

The application of the time-variant discount rate $\bar{m}(t)$ to the investment settings described by Company B results in the statistics shown in Figure 7-11 for the heterogeneous cash flows and Figure 7-12 for the single-payment cash flows. The detailed figures are summarized in Tables A-7 and A-8.

For the heterogeneous cash flows (Case B-I), if Figure 7-11 is compared with Figure 7-5, it is seen that the maximum horizon value for each criterion increases 2.2% for the PB criterion (δ = .5), 16.8% for the expected present worth criterion, -12% for the M-V criterion (λ = .0002), 35.8% for the expected utility criterion, and 19.28% for the local optimum approach. Since the global upper bound solution is not a function of interest rate, no change is observed (see Section 5.3.4.2). These results support the argument made in Section 7.4.1.1 that the PB criterion is not sensitive to the changes in the discount rate for this type of investment setting. Therefore, no change in the efficient frontier curve of the PB criterion is observed.

The application of the time-variant discount rate has seen an

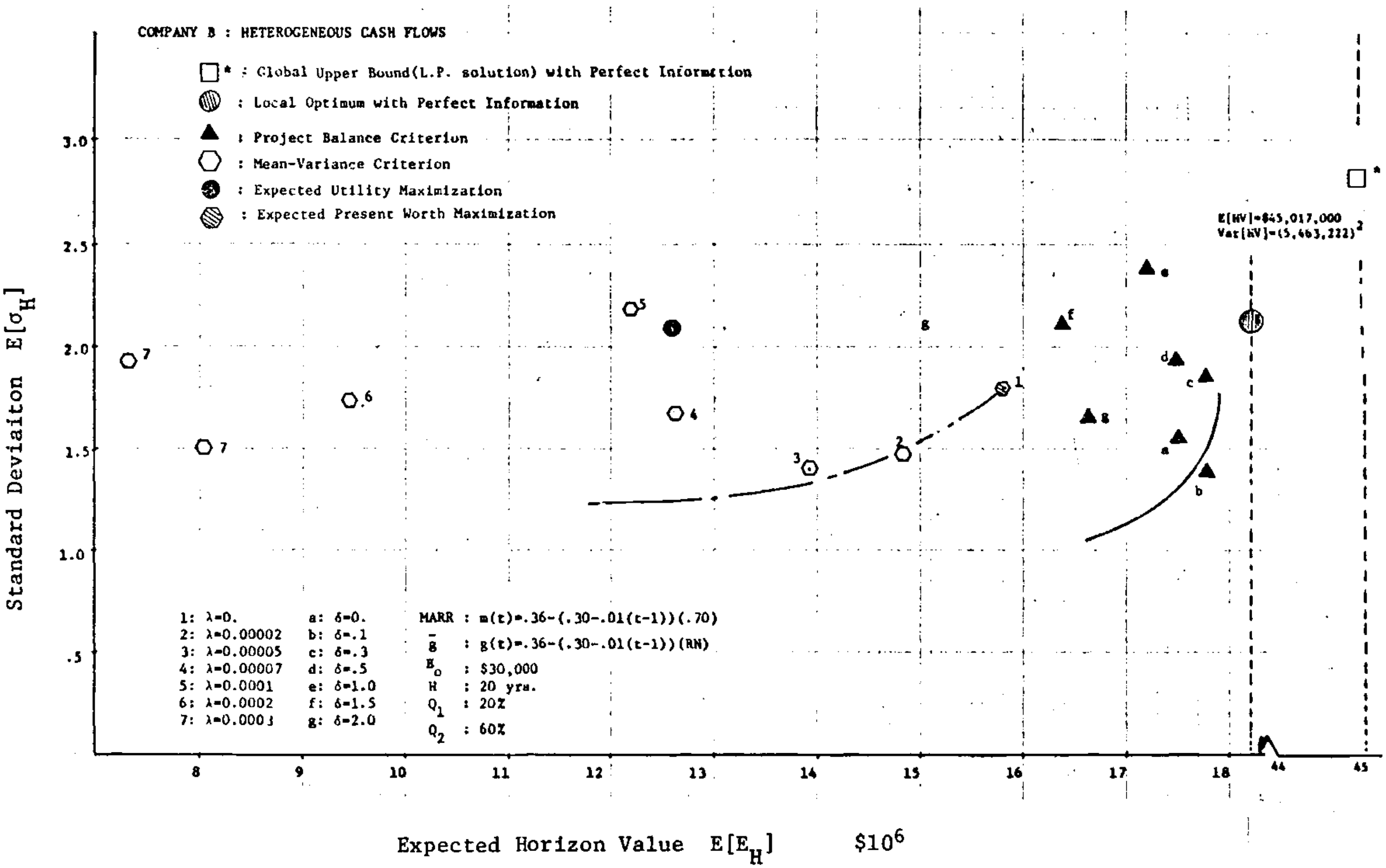


Figure 7-11. Risk-Return Chart: Effects of Time Variant Discount Rate (Company B)--Heterogeneous

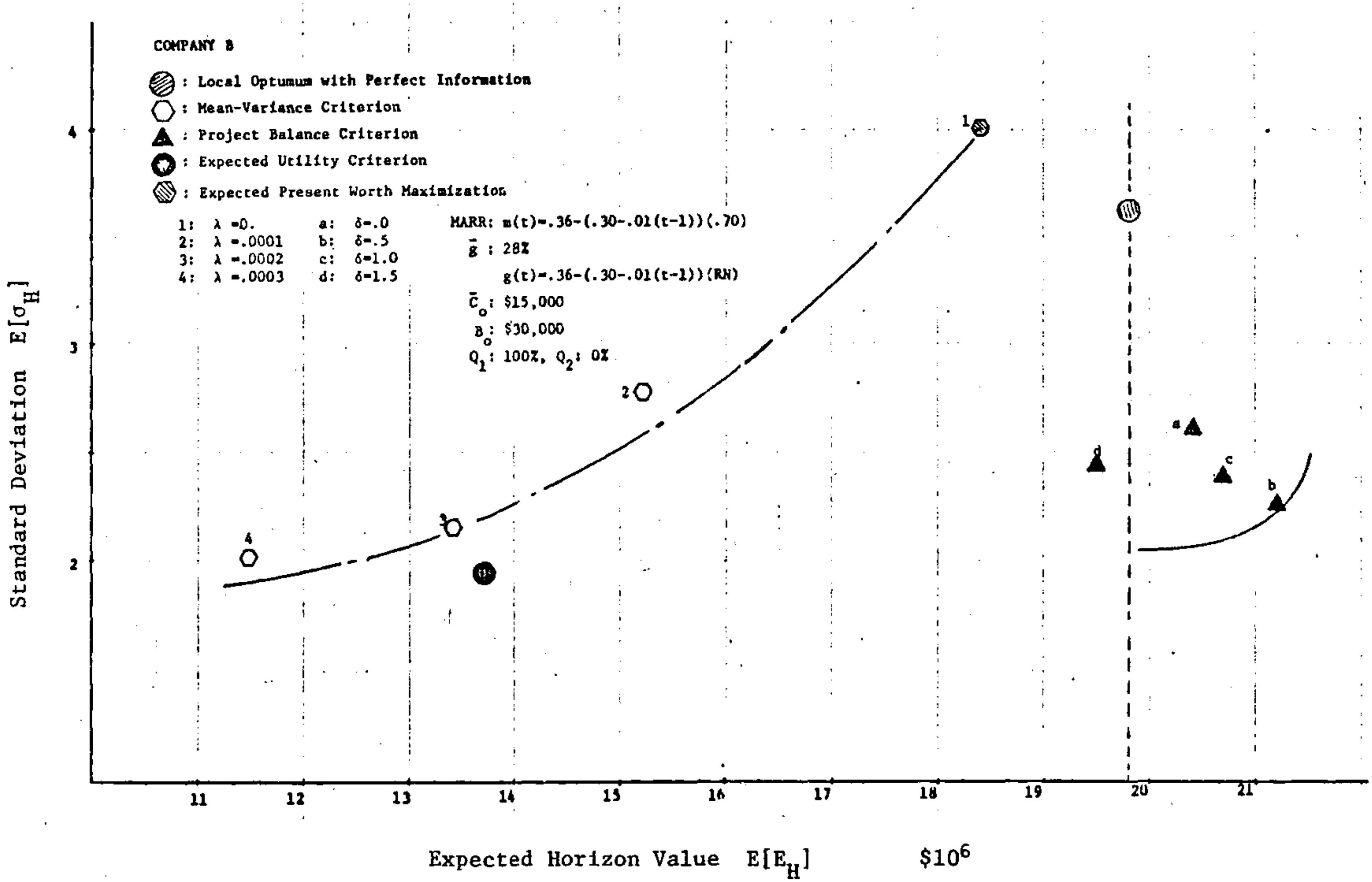


Figure 7-12. Risk-Return Chart: Effects of Time Variant Discount Rate (Company B)--Single Payment

improvement in the performance (in E_H) of both expected present worth and expected utility criteria without increasing any variability appreciably. It is also seen that the application of the discount rate substantially increases the expected horizon value for the local optimum. Despite the substantial improvement made in the performance of the other criteria, the PB criterion consistently yields the selection of proposals that result in greater future value at the horizon time than for any of the other criteria tested.

For the single-payment cash-flow case, Figure 7-12 can be directly compared with Figure 7-6. In terms of both expected horizon value and variability (E_H and σ_H), the improved performance of the PB criterion over other decision criteria is confirmed. By comparing Tables A-8 and A-4, the maximum E_H value increases to 25% for the PB criterion (δ = .5), 19.6% for the expected present worth criterion, 8.4% for the M-V criterion (λ = .0002), 19.8% for the expected utility criterion, and 20% for the local optimum approach.

CHAPTER VIII

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

The primary purpose of this research is to investigate the significance of the concept of the resolution of uncertainty in capital allocation decision problems. Of particular interest are how to measure the concept of the resolution of uncertainty and then how to incorporate this concept in a multi-stage decision process. A complete summary of the results of the research is given in Section 8.1, followed by conclusions in Section 8.2, and recommendations for future research in Section 8.3.

8.1 Summary of Results

This study begins with the discussion of the decision criteria considering risk that have found the widest support in the capital budgeting literature and examines the explicit measures of uncertainty resolution which appeared in the literature. As is evidenced by the review of the literature, it is true that limited attention in the literature is given to the development of a methodology of this concept into the improvement of capital budgeting decisions. Among the three methodologies (the payback period, the coefficient of variation, and certainty equivalent) which deal with the concept of uncertainty resolution in an explict manner, the use of the coefficient of variation approach appeared to have the most potential and practical importance for this study.

However, a close examination of the coefficient of variation reveals that the measure of uncertainty resolution based on the terminal value of a proposal fails to provide complete information on how the uncertainty outcomes are expected to be resolved. It is pointed out that the reason for this deficiency is that the use of the terminal value as a measure of economic desirability does not fully consider the shape of the cash flow pattern over the project's life. The review of the literature indicates that a time-dependent measure of investment worth which reflects the timing and magnitude of probabilistic cash flows be developed so that it becomes possible to measure the rate at which uncertainty about the cash flows is resolved through time.

One of the purposes of this study is to develop the means for measuring the important characteristics of an investment project that would indicate whether the investment would be deemed economically desirable or undesirable. The unrecovered balance method and the project balance method were investigated as a basis for quantifying uncertainty resolution, and the conceptual advantage of the project balance method was observed.

As developed in Chapter III, the project balance pattern provides four different elements of information regarding the desirability of a proposal. They are the area of negative project balance (ANB), the time for the project to recover its inital investment (discounted payback period, Q), the area of positive balance (APB), and the terminal profitability ($S_N(i)$) of the proposal. However, as a basis for measuring uncertainty resolution, only two of these parameters (ANB and APB) appeared to reflect the changes in cash-flow patterns more accurately.

than do the other two parameters.

A particular measure of uncertainty resolution considered in this study was the coefficient of variation approach suggested by Van Horne. In terms of the project balance parameters, Van Horne's measure utilizes the terminal profitability $S_N(i)$ as a basis for quantifying uncertainty resolution. Therefore, the effectiveness of using the ANB and APB as a basis for measuring uncertainty resolution based on the coefficient of variation was compared to Van Horne's method. The conceptual advantage for using the ANB (or APB) over the $S_N(i)$ was observed. However, it was shown that the use of coefficient of variation as a statistical measure of uncertainty resolution has inherent limitations. By definition, the coefficient of variation uses the standard deviation in its numerator and it therefore does not distinguish between positive and negative variations.

The concept of the expected gain confidence limit criterion is introduced as a means of overcoming the deficiencies associated with the coefficient of variation as measure of variability. In this presentation, the decision maker is assumed to be risk averse. In general, the decision maker who is risk averse will prefer a proposal whose ANB is smaller or whose APB is larger if other things are equal. Therefore, in terms of variability, the decision maker is concerned more about the variability on the down side for the APB and the variability on the up side for the ANB. Accordingly, two different schemes of expected gain confidence limit criterion are utilized in this study to place confidence limits in the measurement of variabilities of the statistics (ANB and APB).

When ANB and APB are utilized separately as a basis for measuring uncertainty resolution, it was possible to generate two different information elements regarding uncertainty resolution about the same proposal. A measure of uncertainty resolution based upon ANB statistics $EGCL[ANB]_t$ provides information regarding the firm's uncertainty about the recovery of its initial investment through time. On the other hand, a measure of uncertainty resolution based upon APB statistics $EGCL[APB]_t$ generates information about the firm's uncertainty about the profit accumulation through time.

An investment situation is suggested where investment decisions are made on a regular periodic basis and the objective is to maximize the total accumulated wealth at some horizon time. It is assumed that knowledge about what investment would be available in the future and their associated cash flows is probabilistic.

To utilize information about the resolution of uncertainty over time and assess the effect of investment proposals that have probabilistic outcomes for the investment situations described above, a decision criterion called the Project Balance (PB) Criterion is developed. In the PB criterion, a single index is devised as an operational decision rule to seek a practical trade-off among the three major investment factors such as profitability, variability and flexibility. Since both resolution indexes, $EGCL[ANB]_t$ and $EGCL[APB]_t$, represent time-dependent measures of uncertainty resolution, the use of information derived from these resolution indexes in the development of the PB criterion is discussed. The conceptual advantage using $EGCL[ANB]_t$ over $EGCL[APB]_t$ in the development of the PB criterion is described. In fact, information derived from $EGCL[APB]_t$ is not utilized in the PB criterion. However, the possible use of information generated from $EGCL[APB]_t$ is presented in a different capital budgeting context (for example, incorporation of the abandonment decision).

A simulation model based on these investment situations is developed to test the effectiveness of the PB criterion with those of the most widely suggested decision criteria. These criteria are the expected value maximization, the mean-variance (M-V) crterion, and the expected utility maximization. Due to the practical difficulty associated with implementing the expected utility maximization as a decision criterion, an approximate method is utilized. Therefore, in this study, the purpose of the inclusion of the expected utility criterion is not to draw a firm conclusion on the effectiveness of the criterion as compared to other criteria, but to provide the reader with some inference about the criterion.

Besides comparing these three criteria with the PB criterion, the value of having complete infomation about the future investment opportunities is also introduced to compare the overall effectiveness of the PB criterion. Two different levels of knowledge are examined. The first assumes that the decision maker has complete information about the set of investment proposals that are being considered at the time of decision, while the second use assumes complete information regarding all the proposals being considered throughout the study period. This assumption allows the comparison of the PB criterion with well known deterministic models. Thus, comparison among methods requiring perfect information and methods designed for probabilistic information is possible.

8.2 Conclusions

The primary purpose of the simulation experiments is to answer the following four specific questions.

Q1: How much does knowledge of the manner in which the cash flow uncertainty of investment proposals is resolved over time improve the selection of those proposals?

A1: The Project Balance (PB) criterion is the only criterion which considers the concept of resolution of uncertainty explicitly. It is shown that the performance of the PB criterion is better when the variability of the outcomes increases and the cash-flow patterns are single payment rather than heterogeneous. In other words, this implies that the value of having information regarding the duration of uncertainty of proposals increases, as proposals' uncertainty increases.

Q2: How does the PB criterion perform with respect to the other three criteria (the expected present worth criterion, the mean-variance (M-V) criterion, and the expected utility criterion) ?

A2: General conclusions about the performance of the PB criterion can be drawn from the investment settings described by Company A, Company B, and Company C with respect to other decision criteria.

1. Under the two different assumptions of future investment opportunities, the PB criterion is consistently superior to the other decision criteria examined. However, the effectiveness of the PB criterion as a decision rule is substantially increased when the firm faces growing investment opportunities in the future.
2. For two different sets of cash-flow mixes, the performance of the PB criterion is better when the variability of the outcomes increases

and the cash-flow patterns are single-payment rather than heterogenerous.

3. The M-V criterion is very sensitive to the choice of λ, while the PB criterion is not sensitive to the choice of δ (λ and δ represent the coefficient of risk aversion utilized in each decision criterion). In particular, the use of a relatively high value of λ would significantly reduce the effectiveness of the M-V criterion. It appears that the optimal value of δ lies between 0 and 1.5.

4. The performance of the expected utility criterion is rather discouraging for this particular type of utility function (logarithmic utility function) despite its theorectical appeal as a decision rule under risk. This may be due to the choice of the particular utility function used in this study.

5. The performance of the expected present worth criterion is intriguing. In a relatively variable investment situation (see Figure 7-3), it appears to be a rather good criterion despite its inability to reflect the changes in uncertainty associated with the recovery of investment over time. However, in a relatively stable investment setting (see Figure 7-7), it maintains higher variability than do other decision criteria. The above conclusion represents a trend regarding these outcomes but lack of statistical significance makes it impossible to generalize these results.

Q3: How sensitive are these criteria to changes in the significant parameters associated with a regular periodic decision process?

A3: It is shown that the PB criterion is rather insensitive to changes in the discount rate selected, whereas the selection of the optimal interest rate is critical to the other decision criteria (see Section 7.4.4). This implies that the use of the PB criterion will result in alleviating the decision maker's burden to select "the" optimal interest rate in the evaluation of investment proposals.

Q4: How much can one improve his investment decisions with perfect knowledge of future investment opportunities?

A4: Two different levels of knowledge are examined. The first assumes that the decision maker has complete information about the set of investment proposals that are being considered at the time of decision, while the second assumes complete knowledge regarding all the proposals being considered at both the present and future.

As expected, having complete knowledge regarding all the proposals being considered throughout the study period results in an expected horizon value substantially greater than the one that can be achieved with periodic perfect information. In general, it is observed that the value of perfect information regarding the future investment opportunities increases, whenever the variabilities in the investment proposals increase.

For the two different cash flow mixes, it is observed that the relative value of having perfect information about the future investment opportunities for the single-payment type of cash flow is always greater than in the heterogeneous case. One of the significant observations is that the use of PB criterion could result in total capital accumulated at the horizon time that is greater than or equivalent to that which can be achieved with the complete information about the set of investment proposals that are being considered at the time of decision (see the reason in Section 7.4.3.2).

From the results of the simulation, it is also possible to develop insight into the dynamics of a periodic decision model. Of particular interest is how the timing and size of the cash receipts from a large number of proposals affect the funds available for investment at future decision times. It is these funds that affect the selection of proposals at these future decision times.

8.3 Recommendations for Future Research

A logical extension of this study is the incorporation of the abandonment decision into the PB criterion by utilizing the information ascertained from the area of positive project balance in Chapter III. In particular, the information generated from uncertainty resolution for the positive project balance ($EGCL[APB]_t$) would provide the decision maker with a tracking signal of when to abandon previously implemented projects. The paramount task would be to establish a trade-off between the salvage value at the decision time and the expected cash receipts during the remaining life of the proposal.

The consideration of effects of covariance among proposals and the resulting maximization of future value also would be of particular interest. In accordance with this extension, an investigation of an efficient solution methodology for the PB criterion would be of special interest. Use of an index model (see Sharpe [86]) might be an avenue of future research.

APPENDICES

APPENDIX A

DISCRIMINATING ABILITY OF THE PROJECT BALANCE METHOD

In Section 3.1.2.4, it was shown that for a given cash flow (a single payment cash flow), there will be no loss of information by summarizing the project balance characteristics as ANB and APB. In this appendix, it is shown that a similar argument can be applied to those regular cash flow patterns such as uniform series, gradient series (decreasing), and gradient series (increasing).

1. Uniform Series: Consider a project whose cash flow pattern is described as a uniform series as shown in Figure 3-12 (see also its corresponding project balance pattern in Figure 3-12). Unlike the type of single payment cash flow, the $S_t(i)$ values are decreasing for $0 \leq t \leq Q$ (area of negative project balance) as time passes, and after the break-even point Q, the $S_t(i)$ values (area of positive project balance) are increasing as the project goes forward.

For a proposal whose cash flows are described as a uniform series, the break-even value Q will always exist as long as the rate of return of the proposal is greater than the MARR, i. Since $F_1 = F_2 = \ldots = F_N$, the break-even value would be computed by solving the following polynomial equation:

$$S_q(i) = -(1+i)^q F_0 + (1+i)^{q-1} F_1 + \ldots + F_q \, . \tag{A-1}$$

Let $(1+i) = r$, and $F_1 = F_2 \ldots = F_N = A$, then Equation A-1 becomes

$$S_q(i) = -F_0 r^q + Ar^{q-1} + Ar^{q-2} + \ldots + A \tag{A-2}$$

Then, Q is found by letting $S_q(i) = 0$ and solving Equation A-2,

$$q = \{\ell n[\frac{A}{F_0(1-r)+A}]\}/\ell n(r)$$

However, the cash flows occur at discrete points in time and therefore Q is the largest integer that is less than or equal to q, that is, Q = min[q].

Since the value of Q is known, the values of ANB and APB can be computed from successive applications of the geometric series, as shown in the single payment case.

$$ANB = \sum_{t=0}^{Q} S_t(i) = (\frac{1}{1-r}) \left[-F_o(1-r^{Q+1}) + A \left\{ Q - \frac{r(1-r^Q)}{1-r} \right\} \right] \tag{A-3}$$

$$APB = \sum_{t=Q}^{N} S_t(i) = \frac{r}{1-r} \left[\frac{A(N-Q)}{r} + A \left(\frac{r^N - r^Q}{1-r} \right) + F_o\left(r^N - r^Q \right) \right] \tag{A-4}$$

Now, from Equations A-3 and A-4, for fixed values of i and N, ANB and APB are only functions of F_0 and A. If F_0 and A are given, Q will be uniquely determined such that the values of ANB and APB will also be unique. Again, there will be no loss of information by summarizing the project balance characteristics as ANB and APB.

2. <u>Gradient Series</u>: Consider a project whose cash flow pattern is described as a gradient series (decreasing) as shown in Figure 3-12.

This type of cash flow pattern will generate a similar type of project balance pattern to the one of the uniform series as shown in Figure 3-12 but with a different Q value.

Let A' represent the magnitude of the first cash flow F_1, then

$$F_t = A' - g'(t - 1) \tag{A-5}$$

where g' is a gradient factor. By replacing F_t by $A' - g'(t - 1)$ for all t in Equation A-1, and solving $S_q(i) = 0$, the break-even value Q (=min[q]) will be determined. Then the values of ANB and APB can be expressed as:

$$\text{ANB} = \sum_{t=0}^{Q} S_t(i) = -F_0 \left[\frac{1 - r^{Q+1}}{1 - r}\right] + \frac{A'}{1-r}\left[Q - r\,\frac{1 - r^{Q}}{1 - r}\right]$$

$$- \frac{g'}{1-r}\left[\frac{Q(Q-1)}{2} - \frac{r}{1-r}\left\{(Q-1) - r\left(\frac{1 - r^{Q-1}}{1 - r}\right)\right\}\right] \tag{A-6}$$

$$\text{APB} = \sum_{t=Q}^{N} S_t(i) = \frac{F_0}{1-r}\left[r^{N+1} - r^{Q+1}\right] + \frac{A'}{1-r}\left[(N-Q) + \frac{r}{1-r}(r^{N} - r^{Q})\right]$$

$$+ \frac{g'}{1-r}\left[\frac{Q(Q-1) - N(N-1)}{2} + \frac{r}{1-r}\left\{(N-Q)\right.\right.$$

$$\left.\left. + \frac{r}{1-r}(r^{N-1} - r^{Q-1})\right\}\right] \tag{A-7}$$

From Equations A-6 and A-7, for fixed values of i and N, ANB and APB are only functions of A', g' and F_0. Since these three parameters determine the shape of the cash flow pattern, the values of ANB and APB will also be uniquely determined. A similar argument can be applied to an increasing gradient series. From the foregoing discussion, it can be said that if any cash flow pattern belongs to one of these regular periodic cash flows, there will be no loss of information in summarizing the project balance pattern as ANB and APB.

APPENDIX B

DISCRIMINATING ABILITY--THE PROJECT BALANCE METHOD BASED ON THE COEFFICIENT OF VARIATION AS COMPARED WITH VAN HORNE'S MEASURE

In Section 4.1.1, it was discussed that when the coefficient of variation is selected as a measure of uncertainty resolution, the CV_t statistics based on project balance serve as a more discriminating measure of uncertainty resolution as compared with the CV_t statistics based on terminal value (i.e., Van Horne's measure). However, it was also recognized that it is possible to have some examples where Van Horne's method discriminates better than ANB and APB. This appendix provides the reasons why these latter situations are less likely to occur than the previous situations.

To illustrate the point concerned, consider proposals i and j which can be described as a probability tree as shown in Figure 4-1, respectively. For simplicity, assume that each proposal has the same number of probability trees K and K-1 trees of them are identical with each counterpart in terms of cash flow realizations and the probabilities associated with them. Futhermore, assume that the ANB figures for each proposal are identical so that the APB figures are only of concern.

The basic functional relationship between the APB parameter and the terminal value $S_N(i)$ which is utilized in Van Horne's measure would be expressed as follows:

$$[APB]' + S_N(i) = [APB] \qquad \text{(A-8)}$$

where $[APB]' = \sum_{t=Q}^{N-1} S_t(i)$, that is, the summation of the positive project balance up to N-1 period

Now, assume that the sample space of each component ($[APB]'$ and $S_N(i)$) in Equation A-8 consists of n equally likely discrete events; the number of events in $[APB]'$ by n_1; the number in $S_N(i)$ by n_2. Let $[APB]' = A$, $S_N(i) = B$, and $[APB] = C$. Then, the sample spaces of A, B, and C for proposals i and j may be defined as

$$S_{A,i} = \{1,2,3,\ldots,n_1\},\quad S_{A,j} = \{1,2,3,\ldots,n_2\}$$
$$S_{B,i} = \{1,2,3,\ldots,n_2\},\quad S_{B,j} = \{1,2,3,\ldots,n_2\}$$
$$S_{C,i} = \{2,\ldots,n_1+n_2\},\quad S_{C,j} = \{2,\ldots,n_1+n_2\}$$

For the purpose of simplicity, let $n_1 = n_2 = n_o$. In ordered sampling with two trials (A,B) and n_o elements in the population, there are n_o^2 different events in the sample space, S_C. Therefore, for independent proposals i and j, there would be a total number of $(n_o^2)^2$ possible joint events.

Now let X be a random variable that counts the number of the identical C value in each ordered sample point. That is, the sample space for X would be $\{2,3,4,\ldots,2n_o\}$. The values of X associated with various sample points (S_A, S_B) can be described as follows:

Ordered Sample Set $\{[A = a], [B = b]\}$	Sample Space of $[X = x]$	Total Number of Events $[X = x]$
$[A = 1], [B = 1]$	A,B	
$[A = 1], [B = 2]$	$[X = 2] = \{(1,1)\}$	1
$[A = 1], [B = 3]$	$[X = 3] = \{(1,2),(2,1)\}$	2
. .	$[X = 4] = \{(1,3),(2,2),(3,1)\}$	3
. .	.	.
. .	.	.
$[A = 1], [B = n_o]$	.	.
$[A = 2], [B = 1]$		
$[A = 2], [B = 2]$	$[X = n_o+1] = \{(1,n_o),(2,n_o-1),\ldots\}$	n_o
$[A = 2], [B = 3]$	.	.
. .	.	.
. .	.	.
. .	$[X = 2n_o-1] = \{(n_o-1,n_o),(n_o,n_o-1)\}$	2
$[A = n_o], [B = 1]$	$[X = 2n_o] = \{(n_o,n_o)\}$	1
$[A = n_o], [B = 2]$		
. .		
. .		
. .		
$[A = n_o], [B = n_o]$		

Thus, for a fixed value of n_o, there would be a total number of events $\varepsilon[X = x]$

$$T = \sum_{x=2}^{2n_o} \varepsilon[X = x] = 2[n_o(n_o-1)/2] + n_o = n_o^2$$

It becomes evident that for a paired sample event $\varepsilon[C_i=C_j]$, there would be a total number of joint events Ψ

$$\begin{aligned}\Psi = \Sigma\, \varepsilon[C_i=C_j] &= (1)^2 + (2)^2 + (3)^2 + \ldots + (n_o)^2 \\ &\quad + (n_o-1)^2 + \ldots + (2)^2 + (1)^2 \\ &= [(n_o-1)n_o(2n_o-1)]/6 + n_o^2\end{aligned}$$

Thus, the probability that both APBs for proposals i and j are identical can be computed as

$$\text{Prob}[C_i=C_j] = \Psi/T$$
$$= [2n_o^2 + 3n_o + 1]/6n_o^3 \qquad (n_o > 1)$$

In other words, the probability of $[C_i=C_j]$ represents the chance that the project balance method will indicate that both proposals are identical in terms of uncertainty resolution.

Now it is also possible to compute the probability of $[B_i=B_j]$ because for a paired events, the total number of $\varepsilon[B_i=B_j]$ is simply $n_o(n_o^2)$. Therefore,

$$\text{Prob}[B_i=B_j] = n_o(n_o^2)/(n_o^2)^2$$
$$= 1/n_o \qquad , n_o > 1$$

Then, the total number of joint events $\varepsilon[B_i=B_j$ and $C_i=C_j]$ would be

$$\Pi = \Sigma\ \varepsilon[B_i=B_j \text{ and } C_i=C_j] = n_o^2$$

and the probability of $[B_i=B_j$ and $C_i=C_j]$ is

$$\text{Prob}[B_i=B_j \text{ and } C_i=C_j] = \Pi/T$$
$$= 1/n_o^2 \qquad (n_o > 1)$$

In order to evaluate the probability of $\varepsilon[B_i=B_j]$ but not $\varepsilon[B_i=B_j$ and $C_i=C_j]$, or the probability of $\varepsilon[C_i=C_j]$ but not $\varepsilon[B_i=B_j$ and $C_i=C_j]$, it is necessary to determine the absolute probabilities of exclusive events $\varepsilon[B_i=B_j]$ and $\varepsilon[C_i=C_j]$.

$$|\text{Prob}[B_i{=}B_j]| = \text{Prob}[B_i{=}B_j] - \text{Prob}[B_i{=}B_j \text{ and } C_i{=}C_j]$$

$$= 1/n_o - 1/n_o^2 = (n_o-1)/n_o^2$$

and

$$|\text{Prob}[C_i{=}C_j]| = \text{Prob}[C_i{=}C_j] - \text{Prob}[B_i{=}B_j \text{ and } C_i{=}C_j]$$

$$= [2n_o^2 + 3n_o + 1]/6n_o^3 - 1/n_o^2$$

$$= [2n_o^2 - 3n_o + 1]/6n_o^3 \qquad (n_o > 1)$$

By taking the difference between these absolute probabilities,

$$\Delta = |\text{Prob}[B_i{=}B_j]| - |\text{Prob}[C_i{=}C_j]|$$

$$= [4 - 3/n_o - 1/n_o^2]/6n_o \qquad \text{for } n_o > 1$$

and $\Delta > 0$, which implies that $\text{Prob}[B_i{=}B_j]$ is greater than $\text{Prob}[C_i{=}C_j]$.

APPENDIX C

DETAILED SIMULATION RESULTS--TABLES FOR CHAPTER VII

The detailed simulation results for Chapter VII are tabulated in this appendix. All the figures shown in the tables are expressed in terms of $\$10^6$.

Table A-1. Simulation Results: Company A--Heterogeneous

(all in $\$10^3$)

PROJECT BALANCE CRITERION				MEAN-VARIANCE CRITERION			
δ	E_H	σ_H	F_θ	λ	E_H	σ_H	F_θ
.0	14,158	1,685	6	.0	13,299	1,510	9
.5	13,856	1,514	7	.00005	12,340	1,520	10
1.0	13,837	1,605	8	.0001	11,438	1,552	14
1.5	13,799	1,441	8	.0002	9,222	1,741	22
2.0	13,496	1,455	10	.0003	8,083	1,557	25
2.5	12,697	1,439	13	.0004	7,211	1,357	30
3.0	12,227	1,432	15	.0005	6,586	1,144	33
5.0	8,996	1,692	33				

WITH PERFECT INFORMATION				
LOCAL OPTIMUM			GLOBAL L.P. UPPER BOUND	
E_H	σ_H	F_θ	E_H	σ_H
14,524	1,658	8	26,046	3,552

EXPECTED PW MAXIMIZATION		
E_H	σ_H	F_θ
13,299	1,510	9

EXPECTED UTILITY CRITERION		
E_H	σ_H	F_θ
9,023	1,048	11

MARR :15 %
$\bar{g}$:25 %
i_δ : 6%
$\bar{C}_o$: \$15,000
B_o : \$30,000
Q_1 :20 %, Q_2 :60 %

Table A-2. Simulation Results: Company A--Single Payment

(all in $\$10^3$)

PROJECT BALANCE CRITERION				MEAN-VARIANCE CRITERION			
δ	E_H	σ_H	F_θ	λ	E_H	σ_H	F_θ
.0	16,496	2,497	11	.0	15,421	2,920	20
.5	16,929	2,268	13	.0001	13,427	2,204	23
1.0	17,121	2,557	15	.0002	12,367	1,982	23
1.5	17,484	1,875	14	.0003	11,429	1,504	26
2.0	16,730	1,366	16	.0004	10,213	1,576	28
3.0	16,011	2,759	17	.0005	10,063	1,922	29
4.0	14,977	2,424	22				

WITH PERFECT INFORMATION				
LOCAL OPTIMUM			GLOBAL L.P. UPPER BOUND	
E_H	σ_H	F_θ	E_H	σ_H
16,456	3,240	22	37,982	4,550

EXPECTED PW MAXIMIZATION		
E_H	σ_H	F_θ
15,421	2,920	20

EXPECTED UTILITY CRITERION		
E_H	σ_H	F_θ
11,480	2,403	19

MARR : 15 %
$\bar{g}$: 25 %
i_δ : 6%
$\bar{C}_o$: \$15,000
B_o : \$30,000
Q_1 :100%, Q_2 : 0 %

Table A-3. Simulation Results: Company B--Heterogeneous

PROJECT BALANCE CRITERION				MEAN-VARIANCE CRITERION			
δ	E_H	σ_H	F_θ	λ	E_H	σ_H	F_θ
.0	16,592	1,357	5	.0	13,560	2,263	9
.5	17,197	1,233	6	.0001	12,265	1,582	11
1.0	17,068	1,738	5	.0002	10,719	1,441	15
1.5	16,289	1,811	6	.0003	9,919	1,718	19
2.0	16,416	1,671	6	.0004	9,036	1,678	23
2.5	15,644	1,621	8	.0005	8,248	1,691	26
3.0	14,871	1,414	10	.0006	7,548	1,504	30

EXPECTED PW MAXIMIZATION		
E_H	σ_H	F_θ
13,560	2,263	9

EXPECTED UTILITY CRITERION		
E_H	σ_H	F_θ
9,285	1,898	12

WITH PERFECT INFORMATION				
LOCAL OPTIMUM			GLOBAL L.P. UPPER BOUND	
E_H	σ_H	F_θ	E_H	σ_H
15,318	2,653	8	45,017	5,463

MARR : 15 %
$\bar{g}$: 28 %
i_δ : 6%
$\bar{C}_o$: $15,000
B_o : $30,000
Q_1 : 20 %, Q_2 : 60 %

Table A-4. Simulation Results: Company B--Single Payment

(all in $\$10^3$)

PROJECT BALANCE CRITERION				MEAN-VARIANCE CRITERION			
δ	E_H	σ_H	F_θ	λ	E_H	σ_H	F_θ
.0	17,860	2,295	10	.0	14,626	3,322	27
.5	19,206	3,022	12	.0001	13,610	2,597	26
1.0	18,403	2,119	13	.0002	12,964	2,548	27
1.5	18,772	2,854	14	.0003	11,895	2,232	23

EXPECTED PW MAXIMIZATION		
E_H	σ_H	F_θ
14,626	3,322	27

EXPECTED UTILITY CRITERION		
E_H	σ_H	F_θ
11,297	1,739	23

WITH PERFECT INFORMATION				
LOCAL OPTIMUM			GLOBAL L.P. UPPER BOUND	
E_H	σ_H	F_θ	E_H	σ_H
16,384	2,968	26		

MARR : 15 %
$\bar{g}$: 28 %
i_δ : 6%
$\bar{C}_o$: $15,000
B_o : $30,000
Q_1 : 100 %, Q_2 : 0 %
k = 6

Table A-5. Simulation Results: Company C--Heterogeneous

(all in $\$10^3$)

PROJECT BALANCE CRITERION				MEAN-VARIANCE CRITERION			
δ	E_H	σ_H	F_θ	λ	E_H	σ_H	F_θ
.0	4,650	515	6	.0	4,102	582	6
.5	4,550	461	7	.0001	3,758	453	9
.8	4,812	410	7	.0002	3,320	440	13
1.0	4,303	408	11	.0004	3,480	395	17
2.0	4,017	365	20	.0006	2,611	390	21
3.0	3,442	354	35	.0008	3,102	362	25
				.001	2,554	306	36

EXPECTED PW MAXIMIZATION		
E_H	σ_H	F_θ
4,102	582	6

EXPECTED UTILITY CRITERION		
E_H	σ_H	F_θ
3,450	378	18

WITH PERFECT INFORMATION				
LOCAL OPTIMUM			GLOBAL L.P. UPPER BOUND	
E_H	σ_H	F_θ	E_H	σ_H
4,958	412	6	6,231	501

MARR : 10 %
$\bar{g}$: 20 %
i_δ : 6%
$\bar{C}_o$: \$8,000
B_o : \$30,000
Q_1 : 20 %, Q_2 : 60 %
k = 6

Table A-6. Simulation Results: Company C--Single Payment

(all in $\$10^3$)

PROJECT BALANCE CRITERION				MEAN-VARIANCE CRITERION			
δ	E_H	σ_H	F_θ	λ	E_H	σ_H	F_θ
.0	5,091	386	8	.0	5,029	678	14
.5	5,458	482	10	.0003	4,406	660	15
1.0	5,467	523	9	.0006	3,882	541	21
1.5	5,392	561	10	.0007	3,764	528	23
2.0	5,250	477	13	.001	3,322	486	24
				.002	2,795	351	33
				.005	2,248	296	48

EXPECTED PW MAXIMIZATION		
E_H	σ_H	F_θ
5,029	678	14

EXPECTED UTILITY CRITERION		
E_H	σ_H	F_θ
4,202	561	9

WITH PERFECT INFORMATION				
LOCAL OPTIMUM			GLOBAL L.P. UPPER BOUND	
E_H	σ_H	F_θ	E_H	σ_H
5,572	530		8,362	671

MARR : 10 %
$\bar{g}$: 20 %
i_δ : 6%
$\bar{C}_o$: \$8,000
B_o : \$30,000
Q_1 : 100%, Q_2 : 0 %
k = 6

Table A-7. Simulation Results: Effects of Time Variant Discount Rate (Company B)--Heterogeneous

PROJECT BALANCE CRITERION				MEAN-VARIANCE CRITERION			
δ	E_H	σ_H	F_θ	λ	E_H	σ_H	F_θ
.0	17,539	1,553	7	.0	15,839	1,839	10
.1	17,753	1,398	7	.00002	14,867	1,458	12
.3	17,723	1,862	7	.00005	13,936	1,375	16
.5	17,575	1,900	8	.00007	12,706	1,660	19
1.0	17,278	2,477	10	.0001	12,236	2,163	21
1.5	16,425	2,178	13	.0002	9,478	1,739	29
2.0	14,877	2,167	21	.0003	8,014	1,517	33

EXPECTED PW MAXIMIZATION		
E_H	σ_H	F_θ
15,839	1,839	10

EXPECTED UTILITY CRITERION		
E_H	σ_H	F_θ
12,610	2,129	12

WITH PERFECT INFORMATION				
LOCAL OPTIMUM			GLOBAL L.P. UPPER BOUND	
E_H	σ_H	F_θ	E_H	σ_H
18,273	2,127	9	45,017	5,463

MARR : $m(t)=.36-(.30-.01(t-1))(.70)$
$\bar{g}$: 28 %
i_δ : 6%
$\bar{C}_o$: \$15,000
B_o : \$30,000
Q_1 : 20 %, Q_2 : 60 %

Table A-8. Simulation Results: Effects of Time Variant Discount Rate (Company B)--Single Payment

(all in \$$10^3$)

PROJECT BALANCE CRITERION				MEAN-VARIANCE CRITERION			
δ	E_H	σ_H	F_θ	λ	E_H	σ_H	F_θ
.0	20,440	2,643	12	.0	18,449	4,021	23
.5	21,294	2,242	14	.0001	15,259	2,696	28
1.0	20,753	2,378	16	.0002	13,410	2,244	31
1.5	19,570	2,475	17	.0003	11,598	2,023	35

EXPECTED PW MAXIMIZATION		
E_H	σ_H	F_θ
18,449	4,021	23

EXPECTED UTILITY CRITERION		
E_H	σ_H	F_θ
13,756	1,914	23

WITH PERFECT INFORMATION				
LOCAL OPTIMUM			GLOBAL L.P. UPPER BOUND	
E_H	σ_H	F_θ	E_H	σ_H
16,304	3,113	28		

MARR : $m(t)=.36-(.30-.01(t-1))(.70)$
$\bar{g}$: 28 %
i_δ : 6%
$\bar{C}_o$: \$15,000
B_o : \$30,000
Q_1 : 100%, Q_2 : 0 %

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VITA

Chan Seok Park was born in Mockcheon Chungnam, Korea, on March 17, 1946, second son of Bong Seo and Young Hee Park. He was graduated from Jemulpo High School, Incheon, Korea, in 1964. In the Spring of 1965, he began his engineering education at Hanyang University and was graduated in February 1969, with the degree of Bachelor of Science in Ceramic Engineering.

After one year with Asahi Electrical Insulators Manufacturing Company, Nagoya, Japan, as a production engineer, he joined Marubeni-Iida Corporation, Seoul Branch, Korea as assistant general manager. During his employment he gained practical experience in the area of production and quality control, financial control and international import-export trading transactions.

Leaving Marubeni-Iida Corporation to pursue an advanced degree in industrial engineering, he entered the United States in September 1971, and enrolled at the School of Industrial Engineering at Purdue University in February 1972. In May 1973, he was awarded the Degree of Master of Science in Industrial Engineering (Engineering Economics Option).

He entered the doctoral program of the School of Industrial and Systems Engineering at the Georgia Institute of Technology in September 1973. While at Georgia Tech, he held positions as Graduate Research Assistant, and taught courses in Engineering Economy. Requirements for the doctoral program were completed in November 1976. He specialized in the area of economic decision analysis with minors in stochastic processes and management control systems.